FIRE JUNGLE

CAPS-aligned mathematics for South African learners

GRADE 12

Mathematics

TEXTBOOK

Definitions • Worked Examples • Exercises • Full Answers

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by Joseph Matthew Mitchell

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Fire Jungle Grade 12 Mathematics

A CAPS-aligned study textbook for the FET phase

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How to use this book

Definitions. Each topic opens with the key terms you must know and be able to state in an exam.

Key formulas. The formulas and rules for the topic, gathered in one box for quick revision.

Worked examples. Fully solved problems. The right-hand notes explain the reason for every step.

Exercises. Practice questions. Try them before checking the back of the book.

Answers. Complete answers with working are collected in the Answers section at the end.

Digital edition. Scan the QR code on any page to open the interactive version of this textbook, with animated lessons, narration and unlimited generated practice.

> *Tip: work through the worked examples with a pen in hand, then attempt the exercises with the book closed. Use the digital edition for extra randomised practice on any topic you find hard.*

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CHAPTER 3

Finance

3.1 Finding the period n of an investment
3.2 What is an annuity?
3.3 Future value of an annuity
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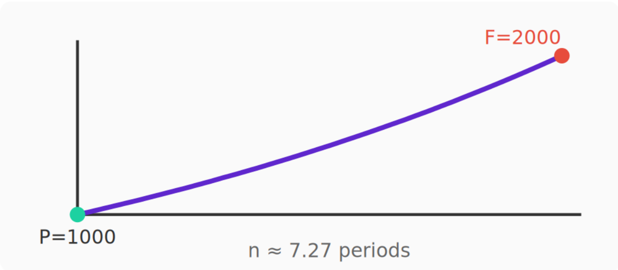
Past Past-paper finance — interest, annuities & a deferred loan

3.1 Period of an investment

Finding the period n of an investment



DEFINITIONS

Compound interest	Interest earned on both the original amount and previously-earned interest.
Period (n)	The number of compounding intervals — what we solve for here.
Logarithm	The inverse of an exponent: if $a^n = b$ then $n = \log b / \log a$. Used to bring n down from the exponent.

KEY FORMULAS

- **Compound growth:** $F = P(1 + i)^n$
- **Solve for n:** $n = \log(F/P) / \log(1 + i)$
- **Doubling:** if $F = 2P$ then $n = \log 2 / \log(1 + i)$

WORKED EXAMPLES

Example 1: R1000 grows to R2000 at 10% compound. Find n.

- $2000 = 1000(1.10)^n$

Write the compound-growth equation.

- $2 = (1.10)^n$

Divide both sides by 1000 to isolate the power.

- $\log 2 = n \cdot \log 1.10$

Take logs of both sides; the exponent comes down.

- $n = \log 2 / \log 1.10 \approx 7.27$ years

Divide. The money doubles in about 7.27 years.

Example 2: R5000 grows to R7500 at 8% p.a. Find n.

- $1.5 = (1.08)^n$

7500/5000 = 1.5.

- $n = \log 1.5 / \log 1.08 \approx 5.27$ years

Take logs and divide.

Applied example — Reaching a savings goal

Thabo invests R12 000 at 9% p.a. compounded monthly. After how many months will it grow to R20 000?

- $20\ 000 = 12\ 000(1 + 0.0075)^n$

Use $A = P(1 + i)^n$ with $i = 0.09/12 = 0.0075$.

- $(1.0075)^n = 1.6667$

Divide both sides by 12 000.

- $n = \log 1.6667 / \log 1.0075 \approx 68.37$ months

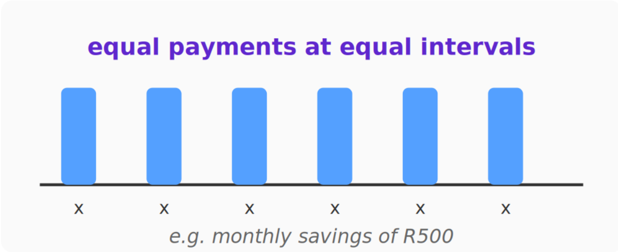
Take logs; ≈ 5.7 years.

EXERCISES (EASY → DIFFICULT)

1. R1 000 grows to R2 000 at 10% p.a. compounded annually. Find n.
2. R5 000 grows to R10 000 at 8% p.a. compounded annually. Find n.
3. R2 000 grows to R3 000 at 12% p.a. compounded annually. Find n.
4. Thabo invests R8 000 at 9% p.a. compounded annually. How long until it is worth R12 000?
5. Naledi wants to double her R15 000 savings at 6% p.a. compounded annually. How many years?
6. Sipho’s R10 000 earns 11% p.a. compounded annually. After how many years is it worth R25 000?
7. Lerato saves R4 000 at 15% p.a. compounded monthly. After how many months does it reach R10 000?
8. An endowment of R20 000 grows at 9% p.a. compounded quarterly. After how many quarters does it reach R50 000?
9. Mrs Naidoo invests R3 000 at 8% p.a. compounded half-yearly. How long to triple her money?
10. Bandile has R30 000 invested at 7,5% p.a. compounded monthly and needs R50 000 for a deposit. After how many months will he have enough?

3.2 Annuities (definition)

What is an annuity?



DEFINITIONS

Annuity	A series of equal payments made at equal time intervals.
Future value annuity	Saving: equal deposits that grow to a lump sum.
Present value annuity	A loan repaid by equal instalments.

KEY FORMULAS

- **Future value:** $F = x[(1 + i)^n - 1] / i$
- **Present value:** $P = x[1 - (1 + i)^{-n}] / i$
- **x** = the equal payment each period

WORKED EXAMPLES

Example 1: Which is an annuity?

▸ Equal payments at equal intervals → annuity.
R500 every month is an annuity.
▸ One single payment is NOT an annuity.
A once-off deposit has no repeating series.

Example 2: Monthly car repayments of R3200 for 60 months.

▸ Same amount, same interval → annuity.
This is a present-value annuity (a loan).

Applied example — Choosing the right formula

Lerato will pay R1 200 into a savings plan at the end of every month for 4 years to build a car deposit. Which annuity formula must she use, and why?

▸ Future-value annuity, $F = x[((1+i)^n - 1)/i]$
She is accumulating regular payments towards a future amount.
▸ Payments are at the end of each period
So it is an ordinary annuity.

EXERCISES (EASY → DIFFICULT)

1. Define an annuity in one sentence.
2. Is a future-value or present-value annuity used to save towards a goal?
3. Which annuity type values a loan or bond?
4. When are payments made in an *ordinary* annuity?

5. When are payments made in an annuity *due*?
6. Lerato saves R500 each month for a house deposit she needs in 5 years. Which annuity formula applies?
7. Thabo takes a home loan repaid monthly over 20 years. Which formula gives the original loan amount?
8. A retirement fund must pay out R8 000 per month for 25 years. Which formula finds the lump sum needed now?
9. A company sets up a sinking fund to replace a R200 000 taxi in 6 years. Which annuity is this?
10. Why does a home loan use the present-value formula?

3.3 Future value annuity formula

Future value of an annuity

Future value annuity

$$\boxed{F = x\cdot[(1+i)^n - 1] / i}$$

DEFINITIONS

Future value (F)	The total accumulated amount after all payments and interest.
Payment (x)	The equal amount deposited each period.
i	The interest rate per period (annual rate ÷ number of periods per year).

KEY FORMULAS

- **Future value:** $F = x[(1 + i)^n - 1] / i$
- **The divisor is i** (the per-period rate)
- **Use when:** saving equal amounts to reach a target

WORKED EXAMPLES

Example 1: Complete the formula $F = x[(1+i)^n - 1] / ?$

▸ The denominator is the per-period rate i.

It comes from summing the geometric series.

▸ Answer: i

$F = x[(1+i)^n - 1] / i$.

Example 2: R500/month, i = 0.01, n = 12. Set up F.

▸ $F = 500[(1.01)^{12} - 1] / 0.01$

Substitute x, i and n into the formula.

Applied example — A 10-year savings plan

Sipho pays R1 500 at the end of each month into a fund earning 8% p.a. compounded monthly. How much will he have after 10 years?

- $i = 0.08/12$, $n = 120$
  Convert the rate to per month and count the payments.
- $F = 1\,500[((1 + i)^{120} - 1)/i]$
  Substitute into the future-value annuity formula.
- $F \approx R274,419.05$
  Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Find the future value of R1 000 deposited yearly for 5 years at 10% p.a. compounded annually.
2. Find the future value of R500 deposited yearly for 10 years at 8% p.a. compounded annually.
3. Find the future value of R2 000 deposited yearly for 8 years at 9% p.a. compounded annually.
4. Naledi deposits R200 at the end of every month for 2 years into an account paying 12% p.a. compounded monthly. How much will she have?
5. Sipho pays R300 a month into a savings plan at 9% p.a. compounded monthly for 3 years. Find the future value.
6. A school saves R5 000 a year at 11% p.a. compounded annually to buy laptops. How much is in the fund after 6 years?
7. Lerato puts away R250 every month for 5 years at 15% p.a. compounded monthly. What is the maturity value?
8. A stokvel invests R1 000 per quarter at 8% p.a. compounded quarterly for 5 years. Find the future value.
9. Thabo needs R100 000 in 4 years and saves monthly at 10,5% p.a. compounded monthly. What monthly deposit is required?
10. A sinking fund must reach R200 000 in 6 years. Payments are monthly at 9% p.a. compounded monthly. Find the monthly payment.

3.4 Present value annuity formula

Present value of an annuity

Present value annuity

$$\boxed{P = x\cdot[1 - (1+i)^{-n}] / i}$$

DEFINITIONS

Present value (P)	The loan amount today that the equal future payments repay.
Negative exponent	$(1+i)^{-n}$ discounts future payments back to today.
Instalment (x)	The equal repayment each period.

KEY FORMULAS

- **Present value:** $P = x[1 - (1 + i)^{-n}] / i$
- **The divisor is i**
- **Use when:** finding a loan amount or instalment

WORKED EXAMPLES

Example 1: Complete $P = x[1 - (1+i)^{-n}] / ?$

▸ The denominator is i.

Same structure as the future-value formula.

▸ Answer: i

$P = x[1 - (1+i)^{-n}] / i$.

Example 2: Instalment R3000, $i = 0.01$, $n = 24$. Set up P.

▸ $P = 3000[1 - (1.01)^{-24}] / 0.01$

Substitute the values.

Applied example — The lump sum a pension needs

A retiree wants to draw R3 000 at the end of each month for 5 years from a fund earning 9% p.a. compounded monthly. What lump sum is needed now?

▸ $i = 0.0075$, $n = 60$

Monthly rate and number of withdrawals.

▸ $P = 3\,000[(1 - (1 + i)^{-60})/i]$

Present-value annuity formula.

▸ $P \approx R144{,}520.12$

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Find the present value of R1 000 a year for 5 years at 10% p.a. compounded annually.
2. Find the present value of R800 a year for 10 years at 8% p.a. compounded annually.
3. Find the present value of R1 500 a year for 6 years at 9% p.a. compounded annually.
4. A car is bought with payments of R400 a month for 2 years at 12% p.a. compounded monthly. What is the cash (present) value?
5. Lerato repays a study loan at R2 000 a year for 12 years at 7% p.a. compounded annually. How much did she borrow?
6. Sipho pays R600 a month for 3 years at 9% p.a. compounded monthly on a furniture account. Find the cash price.
7. A retiree wants R3 000 a month for 5 years from an account paying 15% p.a. compounded monthly. What lump sum is needed now?
8. A bursary pays R1 200 per quarter for 5 years at 8% p.a. compounded quarterly. Find the present value.
9. Thabo can afford R900 a month for 4 years at 10,5% p.a. compounded monthly. What is the largest loan he can take?
10. A pension must pay R8 000 a month for 25 years at 9% p.a. compounded monthly. What lump sum funds it?

3.5 Investment/loan analysis

Converting annual rates to per-period rates

Monthly rate

$$\boxed{i = \text{annual rate} \div 12}$$

DEFINITIONS

Nominal annual rate	The quoted yearly rate before splitting into periods.
Per-period rate (i)	Annual rate ÷ number of compounding periods per year.
Compounded monthly	Interest added 12 times a year, so i = annual/12.

KEY FORMULAS

- **Monthly:** $i = i_{annual} / 12$
- **Quarterly:** $i = i_{annual} / 4$
- **n (months):** years × 12

WORKED EXAMPLES

Example 1: R10000 loan at 12%/yr compounded monthly. Find i.

- ▸ i = 12% / 12
  Divide the annual rate by 12 months.
- ▸ = 1% = 0.01
  The monthly rate is 0.01.

Example 2: 9% p.a. compounded quarterly. Find i.

- ▸ i = 9% / 4 = 2.25% = 0.0225
  4 quarters per year.

Applied example — A home-loan repayment

The Khumalo family borrows R250 000 at 11% p.a. compounded monthly, repaid over 20 years. Find the monthly instalment.

- ▸ $i = 0.11/12$, $n = 240$
  Per-month rate and number of payments.
- ▸ $x = P\cdot i / (1 - (1 + i)^{-n})$
  Rearrange the present-value annuity formula for x.
- ▸ $x \approx$ R2,580.47
  Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Convert 12% p.a. compounded monthly to the rate per month.
2. Convert 9% p.a. compounded quarterly to the rate per quarter.
3. Convert 8% p.a. compounded half-yearly to the rate per half-year.
4. Two banks offer 6% p.a. compounded monthly and 6,1% p.a. compounded annually. Find the effective rate of the first.
5. A bank advertises 10% p.a. compounded quarterly. What effective annual rate does a customer really earn?
6. Find the effective annual rate equivalent to 15% p.a. compounded monthly.
7. Naledi compares 12% compounded monthly with 12,5% compounded annually. Which is better?
8. Sipho takes a R50 000 loan at 9% p.a. compounded monthly over 3 years. Find the monthly repayment.
9. A R100 000 car loan is taken at 12% p.a. compounded monthly over 5 years. Find the monthly instalment.
10. The Khumalo family takes a R250 000 home loan at 11% p.a. compounded monthly over 20 years. Find the monthly repayment.

Past Paper — Finance (NSC P1 Q6)

Past-paper finance — interest, annuities & a deferred loan

DEFINITIONS

Compound interest	Interest earned on the original amount and on the interest already added, given by A equals P times one plus i, to the power n.
Future value annuity	The total value of a series of equal regular deposits, with interest, at the end of the term.
Present value annuity	The amount borrowed today that a series of equal future repayments will exactly settle.
Deferred payment	A loan whose first repayment only starts some months after the money is borrowed, so the balance grows in the meantime.

KEY FORMULAS

- **Compound growth:** $A = P(1 + i)^n$
- **Future value (savings):** $F = x \cdot [(1 + i)^n - 1] \div i$
- **Present value (loan):** $P = x \cdot [1 - (1 + i)^{-n}] \div i$
- **Per period:** i = (annual rate) ÷ (compounding periods per year); n = (periods per year) × (years)

WORKED EXAMPLES

Example 1: NSC P1 · Q6.1 R12 000 was invested at w% p.a., compounded quarterly. After 24 months the value was R13 459. Determine w. ***(4)***

▸ $A = P(1 + i)^n$
Use the compound-interest formula.
▸ $13\ 459 = 12\ 000(1 + w/400)^8$
Quarterly compounding over 24 months gives n = 8 quarters; the quarterly rate is w over 400.
▸ $(1 + w/400)^8 = 1{,}121\ldots$
Divide both sides by 12 000.
▸ $1 + w/400 = \sqrt[8]{(1{,}121\ldots)}$
Take the eighth root of both sides.
▸ $w/400 = 0{,}0144\ldots$ → $\mathbf{w = 5{,}78\%}$
Solve for w and write it as a percentage.

Example 2: NSC P1 · Q6.2 From 31 Jan 2022, R1 000 is deposited at the end of each month at 7,5% p.a. compounded monthly (last deposit 31 Dec 2022). Will there be enough on 1 Jan 2023 for a R13 000 computer? ***(4)***

▸ $F = x \cdot [(1 + i)^n - 1] \div i$
Use the future-value annuity formula.
▸ $i = 0{,}075/12, \quad n = 12$ deposits
Monthly rate; twelve deposits from January to December.
▸ $F = 1000 \cdot [(1 + 0{,}075/12)^{12} - 1] \div (0{,}075/12)$
Substitute the values.
▸ F = R12 421,22
Evaluate the future value.
▸ **No** — R12 421,22 < R13 000 (short by R578,78)
The savings fall short of the price, so he cannot buy the computer.

Example 3: NSC P1 · Q6.3.1 A car costs R250 000; a 15% deposit is paid and the balance is borrowed at 13% p.a. compounded monthly. Calculate the loan. *(1)*

- ▸ Loan = 85% × R250 000

After a 15% deposit, 85% of the price is borrowed.

- ▸ **Loan = R212 500**

Evaluate.

Example 4: NSC P1 · Q6.3.2 The first repayment is 6 months after the loan; it is repaid over 6 years. Calculate the monthly instalment. *(5)*

- ▸ Grow the balance to one month before the first payment: $A = 212\,500(1 + 0{,}13/12)^5$

The first payment is 6 months after the loan, so interest accrues for 5 months first.

- ▸ $A = R224\,262{,}53$

This is the outstanding balance just before repayments begin.

- ▸ $P = x \cdot [\ 1 - (1 + i)^{-n}\ ] \div i$

Use the present-value annuity formula for the repayments.

- ▸ $224\,262{,}53 = x \cdot [\ 1 - (1 + 0{,}13/12)^{-67}\ ] \div (0{,}13/12)$

Payments run from month 6 to month 72 — that is 67 payments.

- ▸ **x = R4 724,96**

Solve for the monthly instalment.

Applied example — Balance still owed on a loan

A R200 000 loan at 10,5% p.a. compounded monthly is repaid over 15 years. How much is still owed after 5 years (60 payments)?

- ▸ $x \approx$ R2,210.80

First find the monthly instalment.

- ▸ Balance = PV of the remaining 120 payments

The outstanding balance is what those future payments are worth now.

- ▸ Balance ≈ R163,841.69

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. R12 000 is invested at 8% p.a. compounded annually. Find its value after 4 years.
2. R5 000 grows to R8 000 at 10% p.a. compounded annually. After how many years?
3. Convert a nominal 9% p.a. compounded monthly to an effective annual rate.
4. Lerato saves R800 a month for 5 years at 9% p.a. compounded monthly. Find the future value.
5. A R250 000 home loan at 12% p.a. compounded monthly is repaid over 20 years. Find the monthly instalment.
6. R15 000 is invested at 7,5% p.a. compounded quarterly for 6 years. Find the value.
7. Thabo's deposit doubles in 9 years under annual compounding. Find the interest rate.
8. How much must be invested now at 8% p.a. compounded monthly to have R100 000 in 10 years?
9. A R200 000 loan at 10,5% p.a. compounded monthly is repaid over 15 years. Find the balance still owing after 5 years (60 payments made).
10. Repayments of R1 500/month start one month after a R120 000 loan at 9% p.a. compounded monthly. How many payments settle it?

End of Chapter 3 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. R1 000 grows to R2 000 at 10% p.a. compounded annually. Find n.
2. Is a future-value or present-value annuity used to save towards a goal?
3. Find the future value of R2 000 deposited yearly for 8 years at 9% p.a. compounded annually.
4. A car is bought with payments of R400 a month for 2 years at 12% p.a. compounded monthly. What is the cash (present) value?
5. A bank advertises 10% p.a. compounded quarterly. What effective annual rate does a customer really earn?
6. R15 000 is invested at 7,5% p.a. compounded quarterly for 6 years. Find the value.
7. Lerato saves R4 000 at 15% p.a. compounded monthly. After how many months does it reach R10 000?
8. A retirement fund must pay out R8 000 per month for 25 years. Which formula finds the lump sum needed now?
9. Thabo needs R100 000 in 4 years and saves monthly at 10,5% p.a. compounded monthly. What monthly deposit is required?
10. A pension must pay R8 000 a month for 25 years at 9% p.a. compounded monthly. What lump sum funds it?

CHAPTER 5

Polynomials

5.1 Difference of two squares
5.2 Degree of a polynomial
5.3 The remainder theorem
5.4 The factor theorem
5.5 Solving a cubic by factoring

5.1 Revision: factor $x^2 - 9$

Difference of two squares

$$x^2 - 9$$

difference of two squares

$(x - 3)$ $(x + 3)$

DEFINITIONS

Difference of squares	$a^2 - b^2$ always factors as $(a - b)(a + b)$.
Perfect square	A number that is something squared, e.g. $9 = 3^2$.

KEY FORMULAS

- **Rule:** $a^2 - b^2 = (a - b)(a + b)$
- **Here:** $x^2 - k^2 = (x - k)(x + k)$

WORKED EXAMPLES

Example 1: Factorise $x^2 - 9$.

▸ $9 = 3^2$, so this is $x^2 - 3^2$.

Recognise the difference of squares.

▸ $= (x - 3)(x + 3)$

Apply $a^2 - b^2 = (a - b)(a + b)$.

Example 2: Factorise $x^2 - 25$.

- $25 = 5^2$

Identify the square.

- $= (x - 5)(x + 5)$

Apply the rule.

Applied example — Area as a difference of squares

A square tabletop of side x m has a square tile of side 3 m cut from a corner. Write the remaining area as a product.

- Area $= x^2 - 9$

Square area minus the removed square.

- $= (x - 3)(x + 3)$ m^2

Difference of two squares.

EXERCISES (EASY → DIFFICULT)

1. Factorise $x^2 - 9$.
2. Factorise $x^2 - 25$.
3. Factorise $4x^2 - 9$.
4. Factorise $9x^2 - 16$.
5. Factorise $25x^2 - 64y^2$.
6. A square garden of side x m has a square pond of side 3 m removed. Write the remaining area as a product of two factors.
7. A square steel plate of side 2x has a square hole of side 5 cut out. Factorise the remaining area $4x^2 - 25$.
8. A square paving block has side x^2. A 4×4 square is removed. Factorise the leftover area $x^4 - 16$ fully.
9. Two square plots have areas 50 m^2 and $2x^2$ m^2. Factorise their difference $50 - 2x^2$.
10. A square room of side $(x + 1)$ has a 3×3 store-room removed. Simplify and factorise $(x + 1)^2 - 9$.

5.2 Cubic polynomials: degree

Degree of a polynomial

$$f(x) = \boxed{3x^3} + 2x^2 - x + 5$$

degree = highest exponent = 3

DEFINITIONS

Degree	The highest power of x that appears in the polynomial.
Cubic	A polynomial of degree 3, e.g. $3x^3 + 2x^2 - x + 5$.
Leading term	The term with the highest power — it determines the degree.

KEY FORMULAS

- **Degree** = the highest exponent of x
- **Linear** = 1, **Quadratic** = 2, **Cubic** = 3

WORKED EXAMPLES

Example 1: Degree of $3x^3 + 2x^2 - x + 5$.

▸ Highest power of x is x^3.
Scan for the biggest exponent.
▸ Degree = 3
It is a cubic.

Example 2: Degree of $7x^5 - x^2$.

▸ Highest power is x^5.
Degree = 5.

Applied example — Degree of a height model

A ball's height is $h(t) = -5t^2 + 20t$. State the degree of the model and what its graph looks like.

▸ Highest power of t is 2
So the degree is 2.
▸ A parabola opening downwards
Because the leading coefficient is negative.

EXERCISES (EASY → DIFFICULT)

1. State the degree of $f(x) = 3x^3 + 2x^2 - x + 5$.
2. State the degree of $g(x) = 7 - 4x$.
3. State the degree of $h(x) = x^5 - x^2 + 9$.
4. What is the leading coefficient of $f(x) = -2x^3 + x - 6$?
5. State the constant term of $f(x) = x^3 - 4x^2 + 7x - 1$.
6. The height of a thrown ball is $h(t) = -5t^2 + 20t$. State the degree and the shape of its graph.
7. A company's profit is modelled by $P(x) = 2x^3 - 5x^2 + 3$. What is the degree, and how many turning points can the graph have?

8. The volume of a box is $V(x) = (x^2 + 1)(x^3 - x)$. What is the degree of V?
9. A water level is modelled by a cubic. What is the greatest number of times it can change between rising and falling?
10. A population model is $f(x) = (2x - 1)^4$. State its degree.

5.3 Remainder theorem

The remainder theorem

substitute $x = 1$

$$f(1) = 1^3 - 2(1) + 1$$

$$= 0$$

remainder 0 ⟹ it is a factor

DEFINITIONS

Remainder theorem	When $f(x)$ is divided by $(x - a)$, the remainder equals $f(a)$.
Substitution	Replacing x with the value a to evaluate $f(a)$.

KEY FORMULAS

- **Remainder theorem:** remainder of $f(x) \div (x - a) = f(a)$
- **So:** just evaluate $f(a)$

WORKED EXAMPLES

Example 1: Remainder of $f(x)=x^3-2x+1 \div (x-1)$.

▸ Remainder = $f(1)$
Use the remainder theorem with $a = 1$.
▸ $f(1) = 1 - 2 + 1 = 0$
Substitute and evaluate.

Example 2: Remainder of $f(x)=x^2+3 \div (x-2)$.

▸ $f(2) = 4 + 3 = 7$
Remainder is 7.

Applied example — Remainder gives a value

A cost model is $C(x) = x^3 - 6x + 4$ (in R1000s). Use the remainder theorem to find the cost when $x = 2$.

▸ Remainder on division by $(x - 2) = C(2)$
The remainder theorem.
▸ $C(2) = 8 - 12 + 4 = 0$
So the cost is R0 (break-even) at $x = 2$.

EXERCISES (EASY → DIFFICULT)

1. Find the remainder when $f(x) = x^3 - 2x + 1$ is divided by $(x - 1)$.
2. Find the remainder when $f(x) = x^3 + x - 3$ is divided by $(x - 2)$.
3. Find the remainder when $f(x) = 2x^3 - x^2 + 4$ is divided by $(x - 1)$.
4. Find the remainder when $f(x) = x^3 - 4x^2 + 2$ is divided by $(x + 1)$.
5. Find the remainder when $f(x) = x^4 - 3x + 5$ is divided by $(x - 2)$.
6. A profit model is $P(x) = x^3 - 6x + 4$ (in R1000s). Using the remainder theorem, find the remainder when $P(x)$ is divided by $(2x - 1)$, i.e. evaluate $P(½)$.
7. When a cost polynomial $C(x) = 3x^3 + 2x^2 - 5$ is divided by $(x + 2)$, what is the remainder?
8. $f(x) = x^3 + kx - 6$ gives remainder 0 when divided by $(x - 2)$. Find k.
9. $f(x) = x^3 - 3x^2 + ax + 2$ leaves remainder 4 on division by $(x - 1)$. Find a.
10. $x^3 + px^2 + qx + 3$ leaves remainder 6 when divided by $(x - 1)$ and remainder 2 when divided by $(x + 1)$. Find p and q.

5.4 Factor theorem

The factor theorem

substitute x = 2

$f(2) = 2^2 - 4$

= 0

remainder 0 ⟹ it is a factor

DEFINITIONS

Factor theorem	$(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$.
Factor	An expression that divides $f(x)$ exactly, with no remainder.

KEY FORMULAS

- **Factor theorem:** $(x - a)$ is a factor $\iff f(a) = 0$
- **Test:** substitute a; if you get 0, it's a factor

WORKED EXAMPLES

Example 1: Is $(x - 2)$ a factor of $x^2 - 4$?

▸ Test $f(2) = 4 - 4 = 0$

Substitute $a = 2$.

▸ $f(2) = 0 \implies (x - 2)$ IS a factor

Yes.

Example 2: Is $(x - 1)$ a factor of $x^2 + 1$?

▸ $f(1) = 1 + 1 = 2 \neq 0$

Not a factor.

Applied example — Factor theorem for a box

A box has volume $V(x) = x^3 + 3x^2 + 2x$. Show $(x + 2)$ is a factor and factorise fully.

- ▸ $V(-2) = 0$

  Since $V(-2) = 0$, $(x + 2)$ is a factor.
- ▸ $V(x) = x(x + 1)(x + 2)$

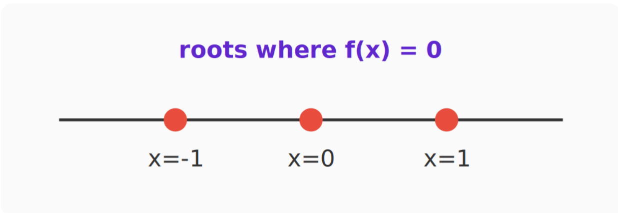
  Take out x, then factorise the quadratic.

EXERCISES (EASY → DIFFICULT)

1. Is $(x - 2)$ a factor of $f(x) = x^2 - 4$?
2. Is $(x - 1)$ a factor of $f(x) = x^3 - 1$?
3. Is $(x + 1)$ a factor of $f(x) = x^3 + 1$?
4. Is $(x - 2)$ a factor of $f(x) = x^3 + x - 3$?
5. Find k if $(x - 1)$ is a factor of $f(x) = x^3 + kx^2 - 4x + 1$.
6. A box has volume $V(x) = x^3 + 3x^2 + 2x$. Show that $(x + 2)$ is a factor.
7. Hence factorise the box volume $V(x) = x^3 + 3x^2 + 2x$ completely.
8. Find a if $(x + 2)$ is a factor of the polynomial $f(x) = 2x^3 + ax + 4$.
9. A tank's volume is $V(x) = x^3 - 2x^2 - 5x + 6\ m^3$. Given $(x - 1)$ is a factor, factorise V fully.
10. Both $(x - 1)$ and $(x + 1)$ are factors of $x^3 + bx^2 + cx - 1$. Find b and c.

5.5 Solving cubic equations

Solving a cubic by factoring



DEFINITIONS

Common factor	A factor shared by all terms — take it out first.
Zero-product rule	If a product = 0, then at least one factor = 0.
Root	A value of x that makes $f(x) = 0$.

KEY FORMULAS

- **Step 1:** take out a common factor
- **Step 2:** factor what remains (often difference of squares)
- **Step 3:** set each factor = 0

WORKED EXAMPLES

Example 1: Solve $x^3 - x = 0$.

- $x(x^2 - 1) = 0$

  Take out the common factor x.
- $x(x - 1)(x + 1) = 0$

  Difference of squares on $x^2 - 1$.
- $x = 0, 1, \text{or } -1$

  Set each factor to zero.

Example 2: Solve $x^3 - 4x = 0$.

- $x(x - 2)(x + 2) = 0$

  Factor out x, then difference of squares.
- $x = 0, 2, \text{or } -2$

  The three roots.

Applied example — Box dimensions from a cubic

An open box has volume $V(x) = x^3 - 4x$ cm³, where x is the width. For which positive width is the volume zero?

- $x(x - 2)(x + 2) = 0$

  Factorise the cubic.
- $x = 2$ cm

  Reject $x = 0$ and $x = -2$ since a width must be positive.

EXERCISES (EASY → DIFFICULT)

1. Solve $x^3 - x = 0$.
2. Solve $x^3 = 8$.
3. Solve $x^3 - 4x = 0$.
4. Solve $(x - 1)(x - 2)(x + 3) = 0$.
5. Solve $x^3 - 3x^2 + 2x = 0$.
6. An open box has volume $V(x) = x^3 - 4x$ (in cm³), where x is its width in cm. For which widths is the volume zero?
7. A container's volume satisfies $x^3 - 6x^2 + 11x - 6 = 0$. Find the possible values of x.
8. Solve $x^3 - 2x^2 - 5x + 6 = 0$ given $x = 1$ is a root.
9. Solve $x^3 + 2x^2 - x - 2 = 0$.
10. Solve $x^3 - 7x - 6 = 0$ given $x = -1$ is a root.

End of Chapter 5 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Factorise $x^2 - 9$.
2. State the degree of $g(x) = 7 - 4x$.
3. Find the remainder when $f(x) = 2x^3 - x^2 + 4$ is divided by $(x - 1)$.
4. Is $(x - 2)$ a factor of $f(x) = x^3 + x - 3$?
5. Solve $x^3 - 3x^2 + 2x = 0$.
6. A square garden of side x m has a square pond of side 3 m removed. Write the remaining area as a product of two factors.
7. A company's profit is modelled by $P(x) = 2x^3 - 5x^2 + 3$. What is the degree, and how many turning points can the graph have?
8. $f(x) = x^3 + kx - 6$ gives remainder 0 when divided by $(x - 2)$. Find k.
9. A tank's volume is $V(x) = x^3 - 2x^2 - 5x + 6$ m³. Given $(x - 1)$ is a factor, factorise V fully.
10. Solve $x^3 - 7x - 6 = 0$ given $x = -1$ is a root.

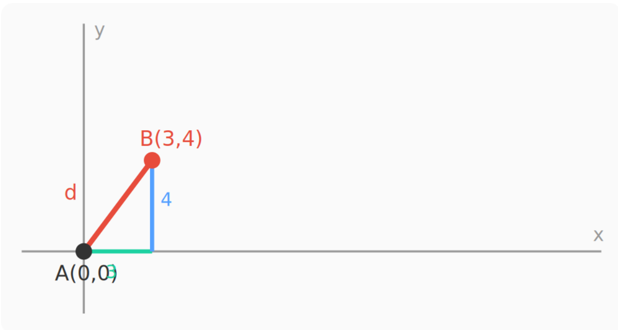
CHAPTER 7

Analytical geometry

7.1 Distance between two points
7.2 Equation of a circle
7.3 Tangent to a circle

7.1 Revision: distance

Distance between two points



DEFINITIONS

Distance formula	$d = \sqrt{[(x_2-x_1)^2 + (y_2-y_1)^2]}$ — from Pythagoras.
Coordinate	An (x, y) pair giving a point's position on the plane.

KEY FORMULAS

- **Distance:** $d = \sqrt{[(x_2-x_1)^2 + (y_2-y_1)^2]}$
- **From origin:** $d = \sqrt{(x^2 + y^2)}$

WORKED EXAMPLES

Example 1: Distance from (0,0) to (3,4).

▸ $d = \sqrt{[3^2 + 4^2]}$

Square each coordinate difference.

▸ $= \sqrt{25} = 5$

The classic 3-4-5 triangle.

Example 2: Distance from (0,0) to (6,8).

- ▸ $d = \sqrt{[36 + 64]} = \sqrt{100} = 10$

A 6-8-10 triangle.

Applied example — Distance on a town map

On a town grid (km), the school is at (2, 3) and the clinic at (5, 7). How far apart are they?

- ▸ $d = \sqrt{[(5-2)^2 + (7-3)^2]}$

Distance formula.

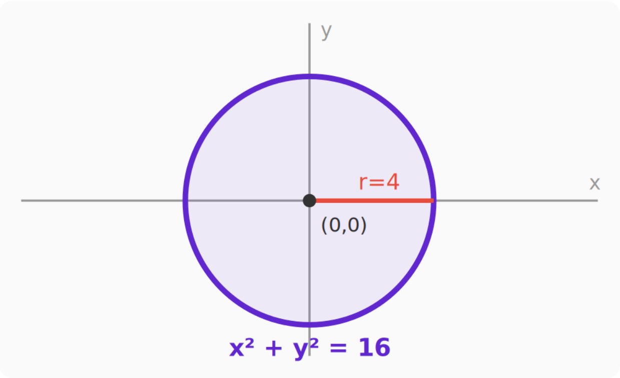
- ▸ $= \sqrt{(9 + 16)} = \sqrt{25} = 5$ km

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Find the distance between (0, 0) and (3, 4).
2. Find the distance between (0, 0) and (6, 8).
3. Find the distance between (1, 2) and (4, 6).
4. Find the distance between (−1, −1) and (2, 3).
5. Find the distance between (2, 3) and (5, 7).
6. On a town map (units in km), the school is at (2, 3) and the clinic at (5, 7). How far apart are they?
7. A drone flies in a straight line from (−2, 1) to (1, 5) (units in km). Find the distance flown.
8. Two cellphone towers stand at (0, 0) and (7, 24) on a grid (units in km). Find the distance between them.
9. A hiker walks from the campsite (3, −2) to a viewpoint (−1, 1) (units in km). How far is the viewpoint?
10. A delivery van goes from depot (−3, 4) to a shop (2, −8) (units in km). Find the straight-line distance.

7.2 Equation of a circle



DEFINITIONS

Circle equation	$(x - a)^2 + (y - b)^2 = r^2$, centre (a,b), radius r.
Centre at origin	When (a,b) = (0,0) the equation simplifies to $x^2 + y^2 = r^2$.
Radius	The distance from the centre to any point on the circle.

KEY FORMULAS

- **General:** $(x - a)^2 + (y - b)^2 = r^2$
- **Centre origin:** $x^2 + y^2 = r^2$

WORKED EXAMPLES

Example 1: Centre (0,0), radius 4.

▸ $x^2 + y^2 = r^2$

Centre at origin form.

▸ $x^2 + y^2 = 16$

$r = 4$, so $r^2 = 16$.

Example 2: Centre (0,0), radius 6.

▸ $x^2 + y^2 = 36$

$r^2 = 36$.

Applied example — A tower's coverage circle

A cellphone tower at the origin covers a radius of 8 km. Write the boundary equation and check whether a house at (6, 8) is covered (km).

▸ $x^2 + y^2 = 64$

Circle, centre origin, radius 8.

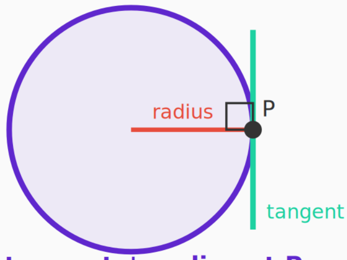
▸ $6^2 + 8^2 = 100 > 64$

The house is 10 km away, so it is outside the coverage.

EXERCISES (EASY → DIFFICULT)

1. Write the equation of a circle centred at the origin with radius 5.
2. Write the equation of a circle centred at the origin with radius 3.
3. State the radius of $x^2 + y^2 = 36$.
4. Write the equation of a circle, centre $(2, -3)$, radius 4.
5. Find the centre and radius of $(x - 1)^2 + (y + 2)^2 = 49$.
6. A cellphone tower at the origin has a signal range of 8 km. Write the equation of its coverage boundary (units in km).
7. A radar at (0, 0) detects aircraft within 10 km. Does a plane at (6, 8) fall inside the range?
8. A sprinkler at point $(2, -3)$ waters a circular area of radius 5 m. Write the equation of the watered region's edge.
9. A circular fish pond is described by $x^2 + y^2 - 6x + 4y - 12 = 0$. Find its centre and radius.
10. A satellite dish must reach a point (5, 12) from a transmitter at the origin (units in km). Write the equation of the circle of that range.

7.3 Tangent to a circle



DEFINITIONS

Tangent	A line that touches the circle at exactly one point.
Point of contact	The single point where the tangent meets the circle.
Key property	The tangent is perpendicular to the radius at the point of contact.

KEY FORMULAS

- **Property:** tangent $\perp$ radius at the point of contact
- **So:** (slope of tangent) $\times$ (slope of radius) $= -1$

WORKED EXAMPLES

Example 1: A tangent at P is perpendicular to the ___?

- ▸ Tangent $\perp$ radius at P

The defining property.

- ▸ Answer: radius

Not the diameter or a chord.

Example 2: Radius has slope 2. Tangent slope?

- ▸ slopes multiply to -1

Perpendicular lines.

- ▸ tangent slope $= -1/2$

Negative reciprocal of 2.

Applied example — Road tangent to a roundabout

A circular roundabout is $x^2 + y^2 = 25$. A straight road touches it at (3, 4). Find the road's equation.

- ▸ Gradient of radius $= 4/3$

From centre (0,0) to (3,4).

- ▸ Tangent gradient $= -3/4$

Perpendicular to the radius.

- ▸ $y - 4 = -\frac{3}{4}(x - 3) \rightarrow y = -\frac{3}{4}x + 25/4$

Point–gradient form.

EXERCISES (EASY → DIFFICULT)

1. A tangent meets a circle in how many points?
2. What is the angle between a tangent and the radius at the point of contact?
3. The radius to the point of contact has gradient 2. What is the tangent's gradient?
4. For the circle $x^2+y^2=25$ and point (3, 4) on it, find the gradient of the radius to (3, 4).
5. Hence find the gradient of the tangent at (3, 4).
6. A circular roundabout is modelled by $x^2 + y^2 = 25$. A straight road touches it at (3, 4). Find the equation of the road (the tangent).
7. The same roundabout $x^2 + y^2 = 25$ has another road touching it at (−3, 4). Find that road's equation.
8. A circular dam $x^2 + y^2 = 10$ has a path touching it at (1, 3). Find the path's equation.
9. Is the straight line $y = -x + 5\sqrt{2}$ a tangent to the circular pond $x^2 + y^2 = 25$? (Use the distance from the centre.)
10. A wheel has centre (2, 1) and the point (5, 5) lies on its rim. Find the gradient of the tangent at (5, 5).

End of Chapter 7 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Find the distance between (0, 0) and (3, 4).
2. Write the equation of a circle centred at the origin with radius 3.
3. The radius to the point of contact has gradient 2. What is the tangent's gradient?
4. Find the distance between (−1, −1) and (2, 3).
5. Find the centre and radius of $(x - 1)^2 + (y + 2)^2 = 49$.
6. A circular roundabout is modelled by $x^2 + y^2 = 25$. A straight road touches it at (3, 4). Find the equation of the road (the tangent).
7. A drone flies in a straight line from (−2, 1) to (1, 5) (units in km). Find the distance flown.
8. A sprinkler at point (2, −3) waters a circular area of radius 5 m. Write the equation of the watered region's edge.
9. Is the straight line $y = -x + 5\sqrt{2}$ a tangent to the circular pond $x^2 + y^2 = 25$? (Use the distance from the centre.)
10. A delivery van goes from depot (−3, 4) to a shop (2, −8) (units in km). Find the straight-line distance.

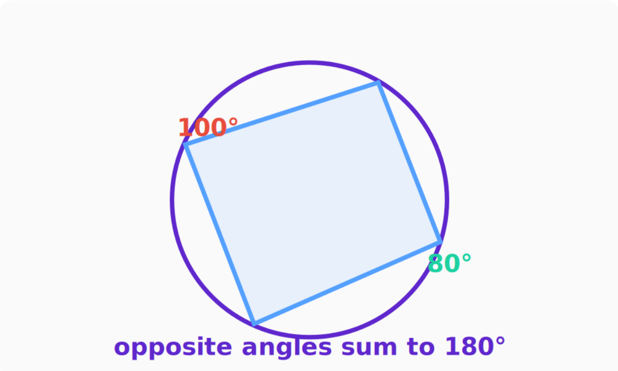
CHAPTER 8

Euclidean geometry

8.1 Cyclic quadrilaterals
8.2 The proportion (intercept) theorem
8.3 Interior angles of a polygon
8.4 Angle sum of a triangle
8.5 Area ratio of similar triangles
8.6 The theorem of Pythagoras

8.1 Revision: cyclic quadrilateral

Cyclic quadrilaterals



DEFINITIONS

Cyclic quadrilateral	A four-sided figure whose vertices all lie on one circle.
Opposite angles	In a cyclic quad, opposite angles add up to 180°.

KEY FORMULAS

- **Cyclic quad:** opposite angles are supplementary
- **So:** angle + opposite angle = 180°

WORKED EXAMPLES

Example 1: One angle is 100°. Opposite angle?

- opposite = $180 - 100$

Opposite angles are supplementary.

- $= 80°$

The opposite angle.

Example 2: One angle is 70°. Opposite?

- $180 - 70 = 110°$

Supplementary.

Applied example — Angles in a round window

A round stained-glass window holds a cyclic quadrilateral ABCD with $\angle A = 88°$. Find $\angle C$.

- $\angle A + \angle C = 180°$

Opposite angles of a cyclic quad are supplementary.

- $\angle C = 92°$

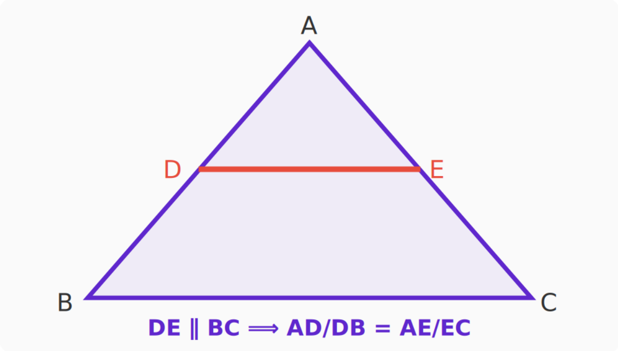
Subtract.

EXERCISES (EASY → DIFFICULT)

1. In a cyclic quadrilateral one angle is 80°. Find the opposite angle.
2. In a cyclic quadrilateral one angle is 95°. Find the opposite angle.
3. Two opposite angles of a cyclic quad are x and 110°. Find x.
4. The angle subtended by a diameter at the circumference equals what?
5. An angle at the centre is 140°. Find the angle at the circumference on the same arc.
6. A round stained-glass window is a cyclic quadrilateral ABCD. $\angle A = 88°$. Find the angle opposite it.
7. Four pillars lie on a circular boundary forming cyclic quad PQRS. $\angle P = 70°$ and $\angle Q = 100°$. Find $\angle R$ and $\angle S$.
8. A Ferris wheel has a horizontal diameter. A cabin sits on the rim, joined to both ends of the diameter. What is the angle at the cabin?
9. Two viewing points on a circular arena look at the same stage arc. One sees it at 38°. What angle does the other see (same segment)?
10. In cyclic quad ABCD, $\angle A = 2x$ and $\angle C = x + 30°$. Find x.

8.2 Ratio and proportion (intercept theorem)

The proportion (intercept) theorem



DEFINITIONS

Proportion theorem	A line parallel to one side of a triangle divides the other two sides in the same ratio.
Parallel line	DE ∥ BC means DE never meets BC and keeps a constant gap.

KEY FORMULAS

- **If DE ∥ BC:** AD/DB = AE/EC
- **The two sides are cut in the same ratio**

WORKED EXAMPLES

Example 1: DE ∥ BC. How are AB and AC cut?

▸ AD/DB = AE/EC

Proportion theorem.

▸ Answer: in the same ratio

Not equal lengths, not perpendicular.

Example 2: AD/DB = 2/3. Then AE/EC = ?

▸ AE/EC = 2/3

Same ratio on both sides.

Applied example — A beam across a truss

On a roof truss, beam DE is parallel to base BC. AD = 2 m, DB = 4 m and AE = 6 m. Find EC.

▸ AD/DB = AE/EC

Proportion theorem (DE ∥ BC).

▸ 2/4 = 6/EC → EC = 12 m

Solve the proportion.

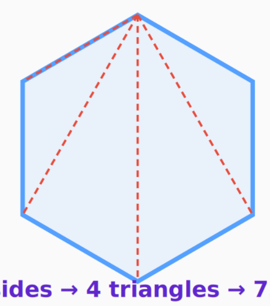
EXERCISES (EASY → DIFFICULT)

1. State the proportion (intercept) theorem in words.
2. DE ∥ BC in △ABC, AD = 4, DB = 2, AE = 6. Find EC.
3. DE ∥ BC, AD = 3, DB = 6, AE = 2. Find EC.
4. If AD/DB = AE/EC, what can you conclude about DE and BC?
5. DE ∥ BC, AD = 5, AB = 8, AE = 10. Find AC.

6. A ladder leans against a wall. A horizontal rung parallel to the floor cuts the two sides so that the upper part is 3 units and lower 6 units on one side, and 2 units on the other upper side. Find the other lower length.
7. On a roof truss, a beam DE is drawn parallel to the base BC of triangle ABC. AD = 2 m, DB = 4 m, AE = 6 m. Find EC.
8. A surveyor draws a line parallel to one side of a triangular plot, splitting side AB in ratio 1:3. In what ratio does it split AC?
9. A midsegment joins the midpoints of two sides of a triangular sail. How does it relate to the third side?
10. In $\triangle ABC$, DE ∥ BC with AD = 2x, DB = x + 2, AE = 6, EC = 4. Find x.

8.3 Sum of interior angles of polygon

Interior angles of a polygon



DEFINITIONS

Interior angle	An angle inside the polygon at a vertex.
Triangulation	An n-sided polygon splits into (n − 2) triangles from one vertex.

KEY FORMULAS

- **Sum of interior angles:** $(n - 2) \times 180°$
- **Because:** the polygon splits into (n − 2) triangles

WORKED EXAMPLES

Example 1: Sum of interior angles of a hexagon (6 sides).

▸ $(n - 2) \times 180 = (6 - 2) \times 180$

Substitute n = 6.

▸ $= 4 \times 180 = 720°$

Four triangles.

Example 2: A pentagon (5 sides).

▸ $(5 - 2) \times 180 = 540°$

Three triangles.

Applied example — Angle of a stop sign

A stop sign is a regular octagon. Find one of its interior angles.

- ▸ Sum = $(8 - 2)\cdot 180° = 1080°$

  Interior-angle sum formula.
- ▸ Each = $1080°/8 = 135°$

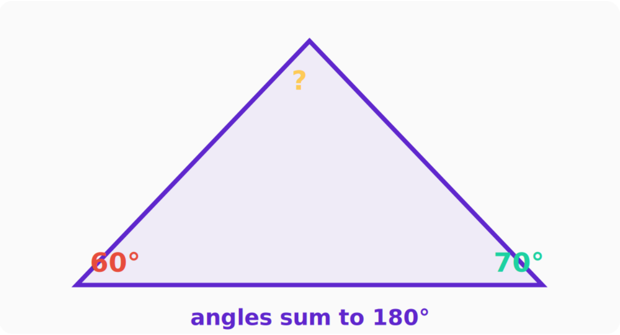
  Divide by the 8 equal angles.

EXERCISES (EASY → DIFFICULT)

1. Find the sum of the interior angles of a triangle.
2. Find the sum of the interior angles of a quadrilateral.
3. Find the sum of the interior angles of a pentagon.
4. Find the sum of the interior angles of a hexagon.
5. Find one interior angle of a regular hexagon.
6. A stop sign is a regular octagon. Find the sum of its interior angles, then one interior angle.
7. A soccer ball panel is a regular pentagon. Find each of its interior angles.
8. A paving tile is a regular polygon whose interior angle is 150°. How many sides has it?
9. A decorative window frame is a regular polygon whose interior angles sum to 1440°. How many sides?
10. Each interior angle of a regular polygon is 108°. How many sides has it?

8.4 Triangles: angle sum

Angle sum of a triangle



DEFINITIONS

Angle sum	The three interior angles of any triangle add up to 180°.

KEY FORMULAS

- **Triangle:** $\text{angle}_1 + \text{angle}_2 + \text{angle}_3 = 180°$
- **Third angle:** $180 - (\text{sum of the other two})$

WORKED EXAMPLES

Example 1: Two angles are 60° and 70°. Third?

- ▸ third = $180 - 60 - 70$

  Angles sum to 180°.
- ▸ $= 50°$

  The third angle.

Example 2: Angles 45° and 55°. Third?

▸ $180 - 45 - 55 = 80°$

Subtract from 180.

Applied example — A roof-gable angle

A roof gable is an isosceles triangle with apex angle 40°. Find each base angle.

▸ Base angles are equal

Isosceles triangle.

▸ $(180° - 40°)/2 = 70°$

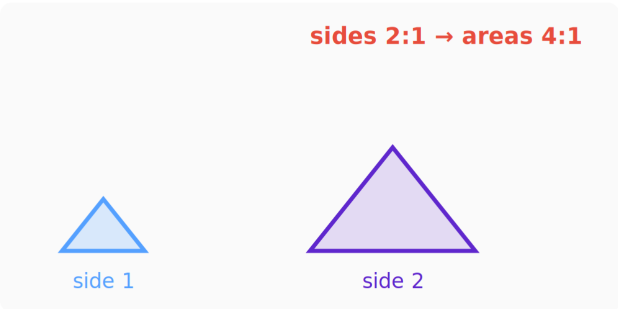
Angles of a triangle sum to 180°.

EXERCISES (EASY → DIFFICULT)

1. Two angles of a triangle are 70° and 60°. Find the third.
2. Two angles are 90° and 35°. Find the third.
3. An equilateral triangle: find each angle.
4. A right-angled isosceles triangle: find the two equal angles.
5. The angles are x, 2x, 3x. Find x.
6. A triangular road sign is isosceles with apex 40°. Find each base angle.
7. A roof gable forms a triangle. An exterior angle at the eaves is 120° and one remote interior angle is 50°. Find the other remote interior angle.
8. A triangular garden has angles in the ratio 2:3:5. Find the largest angle.
9. A bracing strut makes a triangle with angles $(x + 10)$, $(2x - 20)$, $(x + 30)$. Find x.
10. In $\triangle ABC$, $\angle A = \angle B$ and $\angle C = 80°$. Find $\angle A$.

8.5 Similarity (ratio of areas)

Area ratio of similar triangles



DEFINITIONS

Similar triangles	Same shape, sizes related by a constant scale factor k.
Area ratio	If sides are in ratio k:1, areas are in ratio k^2:1.

KEY FORMULAS

- **Sides ratio:** k : 1
- **Areas ratio:** k^2 : 1 (square the side ratio)

WORKED EXAMPLES

Example 1: Similar triangles in ratio 2:1. Area ratio?

- area ratio = 2^2

Square the side ratio.

- = 4 : 1

Areas are 4 times bigger.

Example 2: Side ratio 3:1. Area ratio?

- $3^2 = 9$, so 9 : 1

Square the ratio.

Applied example — Enlarging a sail

A triangular sail of area 12 m^2 is enlarged so each side doubles. Find the new area.

- Area ratio = $(\text{side ratio})^2 = 2^2 = 4$

Areas scale with the square of the side ratio.

- New area = $12 \times 4 = 48$ m^2

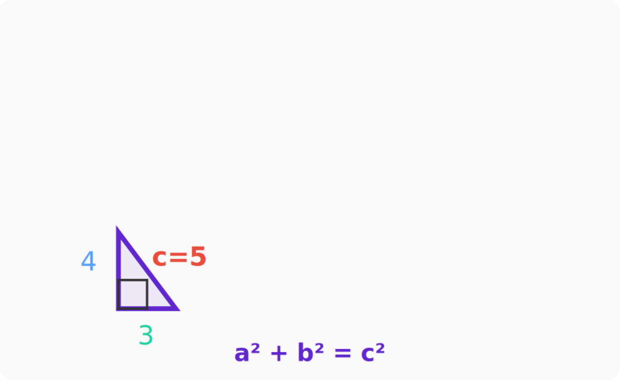
Multiply.

EXERCISES (EASY → DIFFICULT)

1. Two similar triangles have sides in ratio 2:1. Find the ratio of their areas.
2. Two similar triangles have sides in ratio 3:1. Find the ratio of areas.
3. Similar triangles have areas in ratio 16:1. Find the ratio of sides.
4. State the AAA condition for similarity.
5. Sides in ratio 5:2. Find the area ratio.
6. A photo of a triangular sail is enlarged so the sides double. By what factor does its area grow?
7. A small triangular flag has area 12 cm^2. A similar flag has sides twice as long. Find its area.
8. Two similar triangular plots have corresponding sides 6 m and 9 m. The larger has area 81 m^2. Find the smaller area.
9. A scale model of a triangular roof is built at 1:50. Find the ratio of real area to model area.
10. Two similar triangles have perimeters 20 cm and 30 cm. Find the ratio of their areas.

8.6 Pythagorean theorem

The theorem of Pythagoras



DEFINITIONS

Hypotenuse	The longest side of a right triangle, opposite the right angle.
Pythagoras	In a right triangle, $a^2 + b^2 = c^2$ (legs squared sum to hypotenuse squared).

KEY FORMULAS

- **Pythagoras:** $a^2 + b^2 = c^2$
- **Hypotenuse:** $c = \sqrt{a^2 + b^2}$

WORKED EXAMPLES

Example 1: Legs 3 and 4. Hypotenuse?

▸ $c^2 = 3^2 + 4^2 = 9 + 16 = 25$

Sum the squares of the legs.

▸ $c = \sqrt{25} = 5$

Take the square root.

Example 2: Legs 5 and 12. Hypotenuse?

▸ $c^2 = 25 + 144 = 169$

Sum the squares.

▸ $c = 13$

$\sqrt{169} = 13$.

Applied example — Length of a ramp

A wheelchair ramp rises 1,5 m over a horizontal run of 2 m. Find the ramp's length.

▸ $L^2 = 1.5^2 + 2^2 = 6.25$

Theorem of Pythagoras.

▸ $L = 2.5$ m

Take the square root.

EXERCISES (EASY → DIFFICULT)

1. Right triangle, legs 3 and 4. Find the hypotenuse.
2. Right triangle, legs 6 and 8. Find the hypotenuse.
3. Right triangle, legs 5 and 12. Find the hypotenuse.
4. Hypotenuse 13, one leg 5. Find the other leg.
5. Is a triangle with sides 7, 24, 25 right-angled?
6. A 10 m ladder reaches 8 m up a wall. How far is its foot from the wall?
7. A TV screen is 80 cm wide and 60 cm high. Find the diagonal (screen size).
8. A wheelchair ramp rises 1,5 m over a horizontal run of 2 m. Find the length of the ramp surface.
9. A soccer field is 100 m by 75 m. How long is the diagonal a player runs corner-to-corner?
10. A boat sails 9 km east then 12 km north. How far is it from the start?

End of Chapter 8 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. In a cyclic quadrilateral one angle is 80°. Find the opposite angle.
2. DE ∥ BC in △ABC, AD = 4, DB = 2, AE = 6. Find EC.
3. Find the sum of the interior angles of a pentagon.
4. A right-angled isosceles triangle: find the two equal angles.
5. Sides in ratio 5:2. Find the area ratio.
6. A 10 m ladder reaches 8 m up a wall. How far is its foot from the wall?
7. Four pillars lie on a circular boundary forming cyclic quad PQRS. ∠P = 70° and ∠Q = 100°. Find ∠R and ∠S.
8. A surveyor draws a line parallel to one side of a triangular plot, splitting side AB in ratio 1:3. In what ratio does it split AC?
9. A decorative window frame is a regular polygon whose interior angles sum to 1440°. How many sides?
10. In △ABC, ∠A = ∠B and ∠C = 80°. Find ∠A.

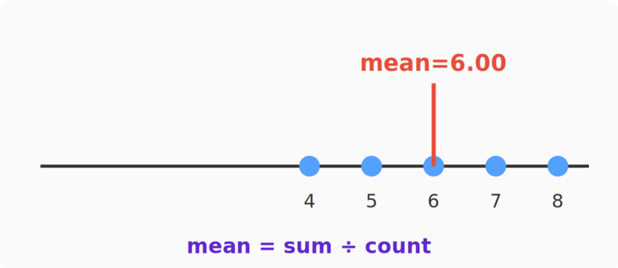
CHAPTER 9

Statistics

9.1 The mean (average)
9.2 The least-squares regression line
9.3 The correlation coefficient r
9.3 Interpreting the value of r

9.1 Revision: mean

The mean (average)



DEFINITIONS

Mean	The average: add all values, then divide by how many there are.
Sum	The total of all the data values added together.

KEY FORMULAS

- **Mean:** $\bar{x}$ = (sum of values) ÷ (number of values)
- **Symbol:** $\Sigma x \div n$

WORKED EXAMPLES

Example 1: Mean of 4, 6, 8, 5, 7.

▸ sum = 4+6+8+5+7 = 30

Add all five values.

▸ mean = 30 ÷ 5 = 6

Divide by the count.

Example 2: Mean of 2, 4, 9.

▸ sum = 15, ÷ 3 = 5

15 over 3 is 5.

Applied example — A batsman's missing score

A batsman averages 42 over 5 innings. After a 6th innings his average is 45. How many runs did he score in the 6th?

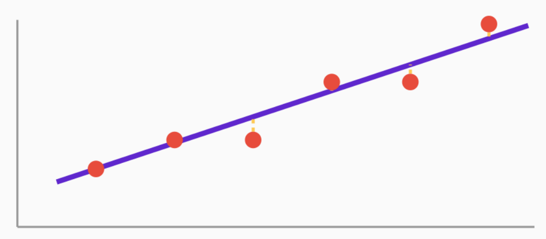
- ▸ Total after 6 = 6 × 45 = 270
  Average × number of innings.
- ▸ Total after 5 = 5 × 42 = 210
  Same for the first five.
- ▸ 6th innings = 270 − 210 = 60 runs
  Subtract.

EXERCISES (EASY → DIFFICULT)

1. Find the mean of 4, 8, 6.
2. Find the mean of 10, 9, 7, 10, 6.
3. Find the mean of 2, 4, 6, 8, 10.
4. Find the mean of 12, 15, 18.
5. Find the mean of 14, 16, 18, 20, 22.
6. Naledi scores 65, 72, 80, 68 and 75 in five tests. Find her mean mark.
7. Rainfall (mm) over a week was 4, 0, 12, 8, 0, 6, 5. Find the mean daily rainfall.
8. A spaza shop's daily sales (R) were 300, 450, 520, 380 and 600. Find the mean.
9. Four learners weigh 50, 55, 60 and x kg with a mean of 56 kg. Find x.
10. A batsman's mean over 5 innings is 42. After a 6th innings his mean is 45. How many runs did he score in the 6th?

9.2 Curve fitting (least squares)

The least-squares regression line



minimise Σ(residuals)² (dashed = residuals)

DEFINITIONS

Least squares	The method that finds the line minimising the sum of squared vertical distances (residuals).
Residual	The vertical gap between a data point and the line.
Best-fit line	The straight line that best summarises the trend in the data.

KEY FORMULAS

- **Goal:** minimise $\Sigma(\text{residual})^2$
- **Residual:** observed y − predicted y

WORKED EXAMPLES

Example 1: What does the best-fit line minimise?

- ▸ It minimises $\Sigma(\text{residuals})^2$

The sum of squared vertical distances.

- ▸ Answer: sum of squared residuals

That is the least-squares criterion.

Example 2: A residual is which distance?

- ▸ vertical distance point→line

Observed minus predicted y.

Applied example — Predicting a test mark

Study time and marks fit $\hat{y} = 30 + 8x$ (x in hours). Predict the mark for 5 hours of study.

- ▸ $\hat{y} = 30 + 8(5)$

Substitute $x = 5$.

- ▸ $\hat{y} = 70\%$

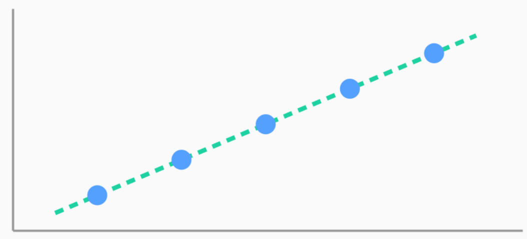
Evaluate. (Avoid predicting for very large x — extrapolation.)

EXERCISES (EASY → DIFFICULT)

1. What does the least-squares regression line minimise?
2. Write the general form of the regression line.
3. In $\hat{y} = a + bx$, what does b represent?
4. A regression line is $\hat{y} = 2 + 3x$. Predict y when $x = 4$.
5. A regression line is $\hat{y} = 10 - 0.5x$. Predict y when $x = 6$.
6. Study time (x hours) versus test mark gives $\hat{y} = 30 + 8x$. Predict the mark for 5 hours of study.
7. Ice-cream sales follow $\hat{y} = 20 + 15x$ where x is the temperature above 20 °C. Predict sales when it is 25 °C ($x = 5$).
8. A regression line always passes through which special point?
9. For a sales model, $\bar{y} = 12$, $\bar{x} = 4$ and $b = 2$. Find the intercept a.
10. Why is it risky to use $\hat{y} = 30 + 8x$ to predict a mark for 20 hours of study?

9.3 Correlation coefficient r

The correlation coefficient r



r = 1: perfect positive

DEFINITIONS

Correlation coefficient r	A number from −1 to 1 measuring how closely points follow a straight line.
r = 1	Perfect positive correlation — all points on a rising line.

KEY FORMULAS

- **Range:** $-1 \le r \le 1$
- **r = 1:** perfect positive · **r = −1:** perfect negative · **r = 0:** none

WORKED EXAMPLES

Example 1: r = 1 means the correlation is?

▸ All points on a rising straight line.

Positive slope, exact fit.

▸ Answer: perfect positive

r = 1.

Example 2: r = 0 means?

▸ No linear relationship.

Points show no straight-line trend.

Applied example — Reading a correlation

A study of hours studied versus marks gives r = 0,9. Describe the relationship.

▸ r is close to +1

Strong positive linear correlation.

▸ More study tends to go with higher marks

But correlation does not prove causation.

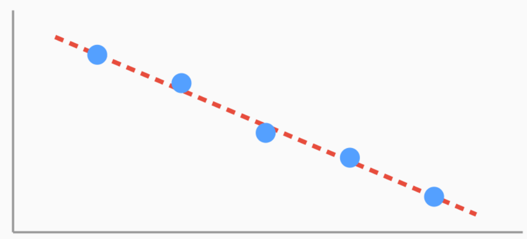
EXERCISES (EASY → DIFFICULT)

1. What range of values can the correlation coefficient r take?
2. What does r = 1 indicate?
3. What does r = −1 indicate?
4. What does r = 0 indicate?
5. r = 0.85 — describe the correlation.
6. A study finds r = 0.9 between hours studied and marks. Describe this relationship.

7. Daily temperature and heater sales give $r = -0.8$. Describe the relationship.
8. Shoe size and test marks for adults give $r = 0.05$. What does this suggest?
9. Which is the stronger correlation: $r = -0.8$ or $r = 0.6$?
10. A report claims $r = 0.7$ proves that x causes y. Is the claim valid?

9.3 Correlation interpretation

Interpreting the value of r



r = −0.9: strong negative

DEFINITIONS

Strong correlation	r close to +1 or −1 — points hug the line tightly.
Negative correlation	As x increases, y decreases (line slopes down).

KEY FORMULAS

- **|r| near 1:** strong · **|r| near 0:** weak
- **Sign:** + rises, − falls

WORKED EXAMPLES

Example 1: $r = -0.9$ indicates?

▸ $|-0.9|$ is close to 1 → strong
Strength from the size.
▸ negative sign → falling
Answer: strong negative correlation.

Example 2: $r = 0.2$ indicates?

▸ $|0.2|$ is small → weak positive
Near 0 means weak.

Applied example — Cause or coincidence?

A town finds r = 0,6 between ice-cream sales and drowning incidents. Explain carefully.

▸ Both rise in hot weather
A lurking variable (temperature) drives both.
▸ Ice-cream does not cause drownings
Correlation ≠ causation.

EXERCISES (EASY → DIFFICULT)

1. $r = 0.95$ — interpret.
2. $r = -0.3$ — interpret.
3. $r = 0.05$ — interpret.
4. Scatter is a random cloud. Estimate r.
5. Rank by strength: $r = -0.7$, $r = 0.5$, $r = 0.95$.
6. As the price of a taxi ride rises, the number of passengers falls steadily. Describe the likely sign and strength of r.
7. A coach finds training hours and goals scored rise together strongly. Estimate r.
8. Can two variables have $r = 0$ yet still be related?
9. A study finds $r = 0.6$ between ice-cream sales and drownings. Explain cautiously.
10. Two car models: fuel use vs engine size gives $r = 0.92$; colour vs price gives $r = 0.04$. Which pairing is linearly related?

End of Chapter 9 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Find the mean of 4, 8, 6.
2. Write the general form of the regression line.
3. What does $r = -1$ indicate?
4. Scatter is a random cloud. Estimate r.
5. Find the mean of 14, 16, 18, 20, 22.
6. Study time (x hours) versus test mark gives $\hat{y} = 30 + 8x$. Predict the mark for 5 hours of study.
7. Daily temperature and heater sales give $r = -0.8$. Describe the relationship.
8. Can two variables have $r = 0$ yet still be related?
9. Four learners weigh 50, 55, 60 and x kg with a mean of 56 kg. Find x.
10. Why is it risky to use $\hat{y} = 30 + 8x$ to predict a mark for 20 hours of study?

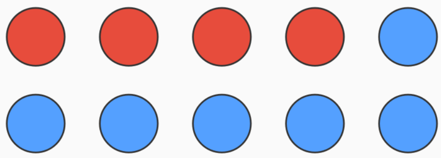
CHAPTER 10

Probability

10.1 Probability of a single event
10.2 The addition rule (inclusion–exclusion)
10.3 Tree diagrams & counting outcomes
10.4 The fundamental counting principle
10.5 Factorial notation
10.6 Permutations (order matters)
10.6 Combinations (order does not matter)
10.7 Probability using counting

10.1 Revision: P of single event

Probability of a single event



P(red) = 4/10

DEFINITIONS

Probability	P(event) = favourable outcomes ÷ total outcomes.
Favourable outcome	An outcome that counts as the event happening.

KEY FORMULAS

- **P(event)** = favourable ÷ total
- **Range:** $0 \le P \le 1$

WORKED EXAMPLES

Example 1: Bag of 10 balls, 4 red. P(red)?

- favourable = 4, total = 10

Count red and total.

- P(red) = 4/10 = 2/5

Favourable over total.

Example 2: Bag of 8, 3 green. P(green)?

- P = 3/8

Three favourable out of eight.

Applied example — Picking a sweet

A bag has 3 red, 2 blue and 5 green sweets. One is taken at random. Find P(green).

- Total = 10 sweets

Add them up.

- P(green) = 5/10 = 1/2

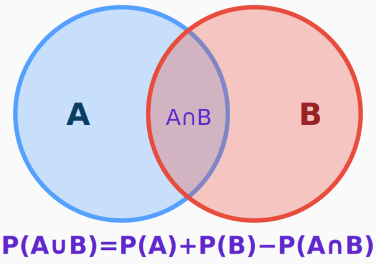
Favourable over total.

EXERCISES (EASY → DIFFICULT)

1. A fair die is rolled. Find P(rolling a 4).
2. A fair coin is tossed. Find P(heads).
3. A die is rolled. Find P(an even number).
4. A card is drawn from 52. Find P(a heart).
5. P(rain) = 0.3. Find P(no rain).
6. A bag of sweets has 3 red, 2 blue and 5 green. A sweet is taken at random. Find P(green).
7. In a class of 30, 12 take Art. One learner is chosen at random. Find P(they take Art).
8. A spinner has numbers 1–8. Find P(a number greater than 5).
9. A box has 12 phones, 4 of which are faulty. One is picked. Find P(not faulty).
10. Two fair coins are tossed. Find P(two heads).

10.2 Probability identities

The addition rule (inclusion–exclusion)



DEFINITIONS

Union $P(A\cup B)$	Probability that A or B (or both) happen.
Intersection $P(A\cap B)$	Probability that both A and B happen.
Inclusion–exclusion	Subtract the overlap once so it is not counted twice.

KEY FORMULAS

- **Addition rule:** $P(A\cup B) = P(A) + P(B) - P(A\cap B)$
- **Mutually exclusive:** $P(A\cap B)=0 \implies P(A\cup B)=P(A)+P(B)$

WORKED EXAMPLES

Example 1: Complete $P(A\cup B) = P(A) + P(B) - ?$

▸ The overlap is counted twice.

Once in P(A), once in P(B).

▸ Subtract $P(A\cap B)$

Answer: $P(A \cap B)$.

Example 2: A and B mutually exclusive. $P(A\cup B)$?

▸ $P(A\cap B) = 0$, so $P(A\cup B)=P(A)+P(B)$

No overlap to subtract.

Applied example — Soccer or netball

In a class, P(soccer) = 0,5, P(netball) = 0,4 and P(both) = 0,2. Find P(at least one).

▸ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Addition rule.

▸ $= 0.5 + 0.4 - 0.2 = 0.7$

Substitute and evaluate.

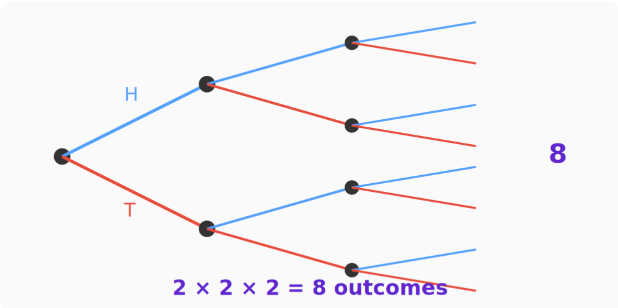
EXERCISES (EASY → DIFFICULT)

1. Complete: $P(A \cup B) = P(A) + P(B) - ?$
2. $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.2$. Find $P(A \cup B)$.
3. For mutually exclusive events, $P(A \cap B) = ?$
4. For independent events, $P(A \cap B) = ?$

5. P(A) = 0.6, P(B) = 0.5, independent. Find P(A ∩ B).
6. In a class, P(plays soccer) = 0.5, P(plays netball) = 0.4, P(both) = 0.2. Find P(plays at least one).
7. 40% of shoppers buy bread and 30% buy milk; 18% buy both. Find the percentage buying bread or milk.
8. Events A and B are mutually exclusive with P(A) = 0.3, P(B) = 0.45. Find P(A ∪ B).
9. A learner passes Maths with probability 0.7 and English with 0.6, independently. Find P(passes both).
10. P(A) = 0.7 and P(A ∩ B) = 0.28 with A, B independent. Find P(B).

10.3 Tools: tree diagram

Tree diagrams & counting outcomes



DEFINITIONS

Tree diagram	A branching diagram showing every possible outcome of a sequence of events.
Outcome	One complete path from start to a leaf of the tree.

KEY FORMULAS

- **Each toss:** 2 branches (H or T)
- **3 tosses:** 2 × 2 × 2 = 8 outcomes

WORKED EXAMPLES

Example 1: Outcomes for 3 coin tosses?

▸ each toss: 2 options

Heads or tails.

▸ 2 × 2 × 2 = 8

Eight leaves on the tree.

Example 2: Outcomes for 2 dice?

▸ 6 × 6 = 36

Each die has 6 faces.

Applied example — Two draws without replacement

A bag has 2 red and 1 blue ball. Two are drawn without replacement. Find P(both red).

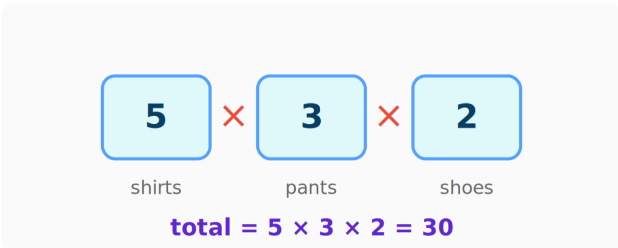
- ▸ P(1st red) = 2/3
  Two of three balls are red.
- ▸ P(2nd red | 1st red) = 1/2
  One red of two balls remains.
- ▸ P(both) = 2/3 × 1/2 = 1/3
  Multiply along the tree branch.

EXERCISES (EASY → DIFFICULT)

1. A coin is tossed 3 times. How many outcomes are there?
2. Toss two coins. List the outcomes.
3. A die is rolled twice. How many outcomes?
4. Toss a coin then roll a die. How many outcomes?
5. A coin is tossed 3 times. Find P(exactly 2 heads).
6. A spinner (red, blue, green) is spun twice. Draw the tree; how many end branches are there?
7. A bag has 2 red and 1 blue ball. Two are drawn without replacement. Using a tree, find P(both red).
8. Same bag, but the first ball is replaced before the second draw. Find P(both red).
9. A learner guesses a 4-question true/false quiz. Find P(all four correct).
10. A robot shows red with P = 0.4, then green with P = 0.6. A car meets two robots. Find P(both red).

10.4 Fundamental counting principle

The fundamental counting principle



DEFINITIONS

Counting principle	If one choice has m options and the next has n, together they give m × n.

KEY FORMULAS

- **Counting principle:** multiply the options at each stage
- **Total** = $n_1 \times n_2 \times n_3 \times \ldots$

WORKED EXAMPLES

Example 1: 5 shirts × 3 pants × 2 shoes. Outfits?

- ▸ multiply the choices

Independent stages.

- ▸ $5 \times 3 \times 2 = 30$

Thirty outfits.

Example 2: 4-digit PIN, digits 0–9. How many?

- ▸ $10 \times 10 \times 10 \times 10 = 10000$

Ten options per digit.

Applied example — Number-plate count

A plate is 3 letters then 3 digits, repeats allowed. How many are possible?

- ▸ $26 \times 26 \times 26$ for letters

26 choices each.

- ▸ $\times\ 10 \times 10 \times 10$ for digits

10 choices each.

- ▸ $= 17\,576\,000$

Multiply (fundamental counting principle).

EXERCISES (EASY → DIFFICULT)

1. 5 shirts and 3 pants. How many outfits?
2. A menu has 4 starters and 6 mains. How many 2-course meals?
3. A PIN uses 4 digits (0–9), repetition allowed. How many PINs?
4. How many 3-letter "words" from 26 letters, repeats allowed?
5. 5 books are placed on a shelf. How many arrangements?
6. A South African number plate is 3 letters then 3 digits, repeats allowed. How many are possible?
7. A canteen offers 3 mains, 4 sides and 2 drinks. How many different meals (one of each)?
8. A 5-character password is 1 letter then 2 digits with no repeated digit. How many are possible?
9. How many even 3-digit numbers are there (no leading zero, repeats allowed)?
10. Five friends queue for a taxi, but two of them insist on standing together. How many orders are possible?

10.5 Factorial notation

4!

$= 4 \times 3 \times 2 \times 1$

= 24

DEFINITIONS

Factorial n!	The product of all whole numbers from n down to 1.
0!	By definition, $0! = 1$.

KEY FORMULAS

- **n! =** $n \times (n-1) \times \ldots \times 2 \times 1$
- **Examples:** $3! = 6$, $4! = 24$, $5! = 120$

WORKED EXAMPLES

Example 1: Compute 4!.

▸ $4! = 4 \times 3 \times 2 \times 1$

Multiply down to 1.

▸ $= 24$

Four factorial is 24.

Example 2: Compute 5!.

▸ $5 \times 4 \times 3 \times 2 \times 1 = 120$

Five factorial.

Applied example — Ordering finishers

In how many different orders can 6 runners finish a race (no ties)?

▸ 6! arrangements

Each position filled by one runner.

▸ $= 720$

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Evaluate 3!.
2. Evaluate 4!.
3. Evaluate 5!.
4. Evaluate 0!.
5. Simplify $6!/4!$.
6. In how many orders can 6 runners finish a race (no ties)?
7. Simplify $7!/5!$.
8. Simplify $(n+1)!/n!$.
9. Evaluate $8!/(6! \cdot 2!)$.
10. Solve $n!/(n-2)! = 20$.

10.6 Counting: permutations nPr

Permutations (order matters)

Permutation

$^{5}P_{2} = 5!/(5-2)! = 5 \times 4$

= 20

DEFINITIONS

Permutation	An arrangement where order matters.
nPr	The number of ways to arrange r items from n: $n!/(n-r)!$.

KEY FORMULAS

- **nPr =** $n! / (n - r)!$
- **Order matters** (AB ≠ BA)

WORKED EXAMPLES

Example 1: Compute $^{5}P_{2}$.

▸ $^{5}P_{2} = 5!/(5-2)! = 5!/3!$

Apply the formula.

▸ $= 5 \times 4 = 20$

Twenty ordered arrangements.

Example 2: Compute $^{4}P_{2}$.

▸ $4 \times 3 = 12$

Twelve arrangements.

Applied example — Awarding medals

Gold, silver and bronze are awarded among 8 athletes. How many ways?

▸ Order matters → use $^{8}P_{3}$

Different medals are distinct positions.

▸ $^{8}P_{3} = 8 \times 7 \times 6 = 336$

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Evaluate $^{5}P_{2}$.
2. Evaluate $^{6}P_{3}$.
3. Evaluate $^{7}P_{2}$.
4. Evaluate $^{4}P_{4}$.
5. Evaluate $^{9}P_{2}$.
6. In how many ways can gold, silver and bronze be awarded among 8 athletes?
7. How many ways can 4 of 10 books be arranged on a shelf?
8. A school elects a head, deputy and secretary from 7 prefects. How many ways?
9. How many 3-letter arrangements (no repeats) can be made from A, B, C, D, E?
10. Solve $^{n}P_{2} = 30$.

10.6 Counting: combinations nCr

Combinations (order does not matter)

Combination

${}^5C_2 = 5!/(2!\cdot 3!) = 120/12$

= 10

DEFINITIONS

Combination	A selection where order does NOT matter.
nCr	The number of ways to choose r from n: $n!/[r!(n-r)!]$.

KEY FORMULAS

- **nCr =** $n! / [r!(n - r)!]$
- **Order doesn't matter** (AB = BA)

WORKED EXAMPLES

Example 1: Compute ${}^5C_2$.

▸ ${}^5C_2 = 5!/(2!\cdot 3!)$

Apply the formula.

▸ $= 120/12 = 10$

Ten selections.

Example 2: Compute ${}^4C_2$.

▸ $24/4 = 6$

Six selections.

Applied example — Choosing a committee

A committee of 4 is chosen from 10 people. How many committees?

▸ Order does not matter → use ${}^{10}C_4$

A committee is just a selection.

▸ ${}^{10}C_4 = 210$

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Evaluate ${}^5C_2$.
2. Evaluate ${}^6C_3$.
3. Evaluate ${}^7C_2$.
4. Evaluate ${}^4C_4$.
5. Evaluate ${}^9C_3$.
6. A committee of 4 is chosen from 10 people. How many committees are possible?
7. How many handshakes occur if 6 people each shake hands once with every other?
8. From 5 men and 4 women, choose 2 men and 2 women for a panel. How many ways?
9. A learner must choose 3 of 8 elective subjects. How many choices are there?

10. Solve ${}^{n}C_2 = 15$.

10.7 Probability with counting

Probability using counting

Lottery

1 favourable ÷ 100 total

= 1/100

DEFINITIONS

Equally likely	Every outcome has the same chance of occurring.
P with counting	Use counting methods to find favourable ÷ total.

KEY FORMULAS

- **P =** favourable outcomes ÷ total outcomes
- **Use counting** to find each number

WORKED EXAMPLES

Example 1: Pick 1 of 100. P(winning)?

▸ 1 favourable out of 100

One winning ticket.

▸ P = 1/100

Favourable over total.

Example 2: 2 winning tickets of 50. P(win)?

▸ 2/50 = 1/25

Simplify.

Applied example — Probability of all boys

A relay team of 3 is chosen at random from 4 boys and 2 girls. Find P(all boys).

▸ Favourable = ${}^{4}C_3 = 4$

Ways to choose 3 boys from 4.

▸ Total = ${}^{6}C_3 = 20$

Ways to choose any 3 from 6.

▸ P = 4/20 = 1/5

Favourable over total.

EXERCISES (EASY → DIFFICULT)

1. Pick 1 winning ticket out of 100. Find P(win).
2. A 4-digit PIN is guessed once. Find P(correct).
3. A card is drawn from 52. Find P(an ace).
4. Choose 2 of 5 learners at random. Find P(a particular pair is chosen).
5. From 3 red and 2 blue marbles, 2 are drawn. Find P(both red).
6. A relay team of 3 is picked at random from 4 boys and 2 girls. Find P(all boys).

7. The letters of MATHS are arranged at random. Find P that the arrangement starts with M.
8. Five learners sit in a row at random. Find P that two best friends sit together.
9. From a bag of 3 red and 3 blue balls, 3 are drawn. Find P(exactly 2 are red).
10. A 5-letter code is formed from A–Z with no repeats. Find P it spells one fixed chosen word.

End of Chapter 10 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. A fair die is rolled. Find P(rolling a 4).
2. $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.2$. Find $P(A \cup B)$.
3. A die is rolled twice. How many outcomes?
4. How many 3-letter "words" from 26 letters, repeats allowed?
5. Simplify $6!/4!$.
6. In how many ways can gold, silver and bronze be awarded among 8 athletes?
7. How many handshakes occur if 6 people each shake hands once with every other?
8. Five learners sit in a row at random. Find P that two best friends sit together.
9. A box has 12 phones, 4 of which are faulty. One is picked. Find P(not faulty).
10. $P(A) = 0.7$ and $P(A \cap B) = 0.28$ with A, B independent. Find $P(B)$.

CHAPTER 7

Exponents

12.E1 Exponents revision (matric prep)
12.E2 Solving exponential equations
12.E3 Surds and rationalising (matric)

12.E1 Exponents revision (advanced laws)

Exponents revision (matric prep)

$$(a^{m})^{n} = a^{m\times n}$$

Watch the exponents combine: m × n.

DEFINITIONS

Base	The number being raised to a power. In 2^3, the base is 2.
Exponent	The number that says how many times to multiply the base by itself. In 2^3, the exponent is 3.
Power	The whole expression "base raised to exponent", e.g. 2^3.
Like bases	Two expressions with the same base. Only with like bases can you add or subtract exponents directly.

KEY FORMULAS

- **All laws together:** $a^m \cdot a^n = a^{m+n}$; $a^m / a^n = a^{m-n}$; $(a^m)^n = a^{mn}$
- **Zero & negatives:** $a^0 = 1$; $a^{-n} = 1/a^n$
- **Rational:** $a^{m/n} = \sqrt[n]{a^m}$
- **Same-base equation:** if $a^x = a^k$, then $x = k$
- **Common matric trick:** rewrite all bases as smallest prime (e.g. $4=2^2$, $9=3^2$, $27=3^3$)
- **Combining different bases:** express as products like $12=2^2\cdot 3$

WORKED EXAMPLES

Example 1: Simplify $4^3 \times 8^2$.

- **Rewrite with the same base:** $4 = 2^2$, $8 = 2^3$.
  Express as powers of 2.
- $(2^2)^3 \times (2^3)^2 = 2^6 \times 2^6$.
  Power-of-power on each.
- $= 2^{12} =$ **4 096**.
  Add exponents (same base).

Example 2: Simplify $(a^2b)^3 / (a^3b^2)$.

- **Numerator:** $(a^2b)^3 = a^6b^3$.
  Apply power to each factor.
- Now: $a^6b^3 / (a^3b^2) = a^{6-3}b^{3-2}$.
  Quotient rule per base.
- $=$ $\mathbf{a^3b}$.
  Done.

Example 3: Simplify $2^{x+1} \cdot 2^{x-1}$.

- **Same base.** Add exponents: $(x+1) + (x-1) = 2x$.
  Algebraic addition.
- Result: $\mathbf{2^{2x}}$.
  Final form.

Applied example — Doubling money each year

An amount doubles every year, so after x years R1 000 becomes $1\,000 \cdot 2^x$. Simplify $1\,000 \cdot 2^x \cdot 2$.

- $= 1\,000 \cdot 2^{(x+1)}$
  Add the exponents (same base).
- $= 2\,000 \cdot 2^x$
  Or write $2 \cdot 2^x = 2^{(x+1)}$.

EXERCISES (EASY → DIFFICULT)

1. Simplify $a^5 \cdot a^3$.
2. Simplify x^7/x^2.
3. Simplify $(a^3)^4$.
4. Evaluate $2^0 + 3^0$.
5. Simplify $(2x^2y)^3$.
6. Evaluate $4^3 \times 8^2$ and write as a power of 2.
7. Simplify $(a^2b)^3/(a^3b^2)$.
8. A bank doubles an amount each year. Write the value of R1 000 after x years, then simplify $1000 \cdot 2^x \cdot 2$.
9. Simplify $9^2 \cdot 27$ as a single power of 3.
10. Simplify $(3 \cdot 2^x)/2^{(x-1)}$.

12.E2 Solving exponential equations

$$2^{x} = 2^{4}$$

Same base $\Longrightarrow$ exponents must be equal.

DEFINITIONS

Exponential equation	An equation with the unknown in the exponent, like $2^x = 16$.
Same-base method	Rewrite both sides as a power of the same base, then equate the exponents.
Logarithm	The inverse of an exponential. Used when bases cannot be matched.
Substitution	Replace a recurring expression (like 2^x) with a new variable (y) to turn the equation into a polynomial.

KEY FORMULAS

- **Same-base method:** if $a^x = a^k$, then $x = k$
- **Logarithm method:** if $a^x = b$, then $x = \log_a(b)$
- **Change-of-base:** $\log_a(b) = \log(b) / \log(a)$
- **Substitution trick:** for $2^{2x} + 3 \cdot 2^x - 4 = 0$, let $y = 2^x$
- **Surd-like:** for $\sqrt{x} = a$, square both sides; for $a^x = b$, take logs

WORKED EXAMPLES

Example 1: Solve $3^x = 81$.

▸ **Rewrite 81 as a power of 3:** $81 = 3^4$.
Same-base method.
▸ $3^x = 3^4 \Longrightarrow \mathbf{x = 4}$.
Equate exponents.

Example 2: Solve $2^x = 7$.

▸ 7 is not a clean power of 2. **Use logs.**
Take $\log_2$ of both sides.
▸ $x = \log_2(7) \approx \mathbf{2.807}$.
Use change-of-base if needed: log(7)/log(2).

Example 3: Solve $4^x - 5 \cdot 2^x + 4 = 0$.

▸ **Substitute** $y = 2^x$. Then $4^x = (2^2)^x = (2^x)^2 = y^2$.
Reduces to a quadratic.
▸ $y^2 - 5y + 4 = 0 \Longrightarrow (y - 1)(y - 4) = 0 \Longrightarrow y = 1$ or $y = 4$.
Standard quadratic.
▸ Back-substitute: $2^x = 1 \to x = 0$; or $2^x = 4 \to$ **$x = 0$ or $x = 2$**.
Two solutions.

Applied example — Bacteria doubling

A colony doubles each hour, starting at 1: the count is 2^t. After how many hours are there 32?

- ▸ $2^t = 32 = 2^5$
  Write 32 as a power of 2.
- ▸ $t = 5$ hours
  Equate the exponents.

EXERCISES (EASY → DIFFICULT)

1. Solve 2^x = 8.
2. Solve 3^x = 81.
3. Solve 2^x = 32.
4. Solve 5^x = 1.
5. Solve 4^x = 8.
6. A bacterial colony doubles each hour, starting at 1. After how many hours are there 32 bacteria? (Solve 2^x = 32.)
7. An investment triples in value as 3^x, where x is in years. Solve 3^x = 10 to find when it is 10× the start (2 dp).
8. Solve 2^(x+1) = 32.
9. Solve 2^(2x) − 5·2^x + 4 = 0.
10. Solve 3^(2x) − 4·3^x + 3 = 0.

12.E3 Surds and rationalising

Surds and rationalising (matric)

√72 → √(36·2) → 6√2

DEFINITIONS

Surd	An irrational root that cannot be simplified to a whole number, like $\sqrt{2}$ or $\sqrt[3]{5}$.
Rationalising	Multiplying top and bottom by something to remove a surd from the denominator.
Conjugate	For $a + \sqrt{b}$, the conjugate is $a - \sqrt{b}$. Their product $a^2 - b$ is rational — useful for rationalising.
Like surds	Surds with the same number under the root sign. They can be added like fractions: $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$.

KEY FORMULAS

- **Surd laws:** $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$; $\sqrt{a} / \sqrt{b} = \sqrt{a/b}$
- **Pull out perfect squares:** $\sqrt{p^2 \cdot q} = p\sqrt{q}$
- **Conjugate trick:** for $1/(a + \sqrt{b})$, multiply by $(a - \sqrt{b})/(a - \sqrt{b})$
- **Rationalising binomial denominators:** uses difference of squares
- $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$

WORKED EXAMPLES

Example 1: Rationalise $1/(2 + \sqrt{3})$.

▸ **Multiply by conjugate:** $(2 - \sqrt{3})/(2 - \sqrt{3})$.
Conjugate is $(2 - \sqrt{3})$.
▸ Numerator: $1 \cdot (2 - \sqrt{3}) = 2 - \sqrt{3}$.
Just write it out.
▸ Denominator: $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = \mathbf{1}$.
Difference of squares.
▸ Result: $\mathbf{2 - \sqrt{3}}$.
Cleanly rationalised.

Example 2: Simplify $\sqrt{50} + \sqrt{18}$.

▸ $\sqrt{50} = \sqrt{25 \cdot 2} = \mathbf{5\sqrt{2}}$. $\sqrt{18} = \sqrt{9 \cdot 2} = \mathbf{3\sqrt{2}}$.
Pull out perfect squares.
▸ Add: $5\sqrt{2} + 3\sqrt{2} = \mathbf{8\sqrt{2}}$.
Same surd, add the coefficients.

Example 3: Rationalise $2/(\sqrt{5} - 1)$.

▸ Conjugate of $(\sqrt{5} - 1)$ is $(\sqrt{5} + 1)$.
Flip the sign.
▸ Multiply: $2(\sqrt{5} + 1) / [(\sqrt{5} - 1)(\sqrt{5} + 1)] = 2(\sqrt{5} + 1) / (5 - 1) = 2(\sqrt{5} + 1)/4$.
Difference of squares: $(\sqrt{5})^2 - 1^2 = 4$.
▸ Simplify: $\mathbf{(\sqrt{5} + 1)/2}$.
Cancel 2 with 4.

Applied example — Side of a square plot

A square plot has area 50 m^2. Express its side length in simplest surd form.

▸ side = $\sqrt{50}$
Side = $\sqrt{\text{area}}$.
▸ $= \sqrt{25 \times 2} = 5\sqrt{2}$ m
Simplify the surd.

EXERCISES (EASY → DIFFICULT)

1. Simplify $\sqrt{50}$.
2. Simplify $\sqrt{48}$.
3. Simplify $\sqrt{8} + \sqrt{18}$.
4. Simplify $\sqrt{2} \cdot \sqrt{8}$.
5. Simplify $\sqrt{12} - \sqrt{27}$.
6. A square has area 50 m^2. Express its side length in simplest surd form.
7. Rationalise $1/\sqrt{2}$.
8. Rationalise $3/\sqrt{3}$.
9. Rationalise $1/(2 + \sqrt{3})$.
10. Simplify $(\sqrt{7} + \sqrt{2})(\sqrt{7} - \sqrt{2})$.

End of Chapter 7 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Simplify $a^5 \cdot a^3$.
2. Solve $3^x = 81$.
3. Simplify $\sqrt{8} + \sqrt{18}$.
4. Evaluate $2^0 + 3^0$.
5. Solve $4^x = 8$.
6. A square has area 50 m^2. Express its side length in simplest surd form.
7. Simplify $(a^2b)^3/(a^3b^2)$.
8. Solve $2^{(x+1)} = 32$.
9. Rationalise $1/(2 + \sqrt{3})$.
10. Simplify $(3\cdot 2^x)/2^{(x-1)}$.

CHAPTER 8

Number Patterns

12.P1 Arithmetic sequences
12.P2 Quadratic sequences
12.P3 Geometric sequences
12.P4 Finite arithmetic series
12.P5 Finite geometric series
12.P6 Infinite geometric series
Past Past-paper number patterns — quadratic sequences

12.P1 Arithmetic sequences

DEFINITIONS

Sequence	An ordered list of numbers, called terms.
Arithmetic sequence	A sequence where each term is found by adding the same fixed number d to the previous term.
Common difference (d)	The fixed number added each time. $d = T_2 - T_1 = T_3 - T_2 = \ldots$
n-th term formula	$T_n = a + (n - 1)d$. Lets you find any term directly without listing them.

KEY FORMULAS

- **Definition:** constant common difference d
- **n-th term:** $T_n = a + (n - 1)d$
- $\mathbf{d = T_{n+1} - T_n}$
- **Two unknowns?** Solve simultaneously

WORKED EXAMPLES

Example: For 3, 7, 11, 15, … find T_{15}.

- $a = 3$, $d = 4$.

Read off.

- $T_{15} = 3 + 14\cdot 4 = \mathbf{59}$.

Apply formula.

Applied example — Seats in a theatre

A theatre has 20 seats in row 1 and 4 more in each next row. How many seats are in row 10?

- $a = 20$, $d = 4$

Arithmetic sequence.

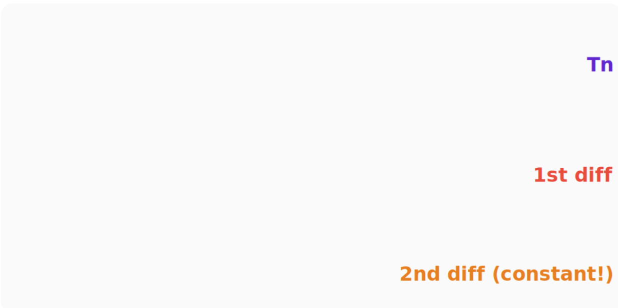
- $T_{10} = 20 + 9(4) = 56$ seats

$T_n = a + (n - 1)d$.

EXERCISES (EASY → DIFFICULT)

1. Find the next term of 3, 7, 11, 15, …
2. Find the common difference of 5, 9, 13, …
3. For 2, 5, 8, … find T_{10}.
4. For 3, 7, 11, … find T_{15}.
5. Find the formula T_n for 1, 6, 11, 16, …
6. A theatre has 20 seats in row 1, 24 in row 2, 28 in row 3, … How many seats are in row 10?
7. Logs are stacked 3 in the top row, 7 in the next, 11 below that, … How many logs are in the 12th row?
8. Lerato's salary starts at R8 000 and rises by R500 each year. What is her salary in year 6?
9. Which term of 4, 7, 10, … equals 100?
10. The 3rd term of an arithmetic sequence is 11 and the 7th is 27. Find a and d.

12.P2 Quadratic sequences



DEFINITIONS

Quadratic sequence	A sequence whose general term is $T_n = an^2 + bn + c$. The SECOND differences are constant.
First differences	The differences between consecutive terms. For a quadratic sequence, these are NOT constant.
Second differences	The differences of the first differences. For a quadratic sequence, these ARE constant and equal to 2a.
General term	A formula T_n that gives any term n directly.

KEY FORMULAS

- **Constant 2nd difference** (not 1st)
- **$T_n = an^2 + bn + c$**
- **$2a$ = constant 2nd difference**

WORKED EXAMPLES

Example: Find T_n formula for 2, 6, 12, 20.

- ▸ 1st diffs: 4, 6, 8. 2nd diff: **2**. So $a = 1$.
  Standard.
- ▸ $T_1 = 1 + b + c = 2$, $T_2 = 4 + 2b + c = 6 \rightarrow b = 1$, $c = 0$.
  Solve.
- ▸ **$T_n = n^2 + n$**.
  Formula.

Applied example — A growing dot pattern

A pattern grows as 2, 6, 12, 20, … dots. Find a formula T_n.

- ▸ First diffs 4, 6, 8; second diff 2
  Quadratic sequence, so $2a = 2 \rightarrow a = 1$.
- ▸ $T_n = n^2 + n$
  Fit b and c using the first terms.

EXERCISES (EASY → DIFFICULT)

1. Find the second difference of 1, 4, 9, 16, …
2. Find a (where $T_n = an^2 + bn + c$) for 1, 4, 9, 16.
3. Find a for 2, 6, 12, 20.
4. Find the next term of 3, 8, 15, 24, …
5. Find T_n for 1, 4, 9, 16, …
6. A pattern of dots forms squares: 1, 4, 9, 16, … dots. How many dots are in the 10th square?
7. Bricks are laid in a triangular stack of 2, 6, 12, 20, … per layer. Find a formula T_n for the nth layer.
8. A quadratic sequence has first differences 4, 7, 10, … and $T_1 = 2$. Find T_4.
9. For 3, 8, 17, 30, … find T_5.
10. For the sequence $T_n = n^2 - 5n + 6$, find n where $T_n = 0$.

12.P3 Geometric sequences

DEFINITIONS

Geometric sequence	A sequence where each term is found by multiplying the previous term by the same fixed number r.
Common ratio (r)	The fixed multiplier. $r = T_2 \div T_1 = T_3 \div T_2 = \ldots$
n-th term formula	$T_n = a \cdot r^{n-1}$.
Decay vs growth	If $\lvert r \rvert < 1$ the terms shrink; if $\lvert r \rvert > 1$ they grow.

KEY FORMULAS

- **Constant ratio r:** $r = T_{n+1}/T_n$
- **n-th term:** $T_n = a \cdot r^{n-1}$

WORKED EXAMPLES

Example: For 2, 6, 18, 54, … find T_6.

▸ $a = 2$, $r = 3$. $T_6 = 2 \cdot 3^5 = 2 \cdot 243 = \mathbf{486}$.

Apply.

Applied example — A bouncing ball

A ball bounces to half its height each time: 16, 8, 4, … m. How high is the 6th bounce?

▸ $a = 16$, $r = ½$

Geometric sequence.

▸ $T_6 = 16(½)^5 = 0.5$ m

$T_n = a \cdot r^{n-1}$.

EXERCISES (EASY → DIFFICULT)

1. Find the next term of 2, 6, 18, …
2. Find the common ratio of 3, 6, 12, …
3. For 1, 5, 25, … find T_5.
4. For 3, 6, 12, … find T_7.
5. Find T_n for 4, 8, 16, …
6. A ball is dropped and each bounce reaches half the previous height: 16, 8, 4, … m. How high is the 6th bounce?
7. A rumour doubles the number of people who have heard it each hour: 2, 4, 8, … After how many terms does it reach 96 from 3? (Which term of 3, 6, 12, … is 96?)

8. A car worth R200 000 loses value to 80% each year: 200 000, 160 000, … Find its value after 3 years (the 4th term).

9. The 2nd term of a geometric sequence is 6 and the 4th is 54 ($r > 0$). Find r.

10. Three numbers x, $x + 3$, $x + 9$ form a geometric sequence. Find x.

12.P4 Finite arithmetic series

2 | 5 | 8 | 11 | 14

$S_n = n/2 \cdot (a + \ell)$

Pairs from each end have the same sum.

DEFINITIONS

Series	The SUM of the terms of a sequence: $T_1 + T_2 + \ldots + T_n$.
Partial sum (S_n)	The sum of the first n terms.
Arithmetic series	The sum of an arithmetic sequence: $S_n = n/2 \cdot (a + \ell)$ where ℓ is the last term.
Last term (ℓ)	The final term to be added: $\ell = a + (n - 1)d$.

KEY FORMULAS

- **Series:** the SUM of a sequence's terms ($S_n = T_1 + T_2 + \ldots + T_n$)
- **Arithmetic series formula:** $S_n = n/2 \cdot (2a + (n - 1)d)$
- **Equivalent form:** $S_n = n/2 \cdot (a + \ell)$, where ℓ is the last term
- **To find n given last term:** use $T_n = a + (n - 1)d$, solve for n
- **To find ℓ:** $\ell = a + (n - 1)d$

WORKED EXAMPLES

Example 1: Find 2 + 5 + 8 + 11 + … + 20.

▸ **a = 2, d = 3.** Last term $\ell = 20$.
Read off.
▸ **Find n:** $20 = 2 + (n - 1)\cdot 3 \rightarrow 3(n - 1) = 18 \rightarrow n = \mathbf{7}$.
Solve $T_n = 20$.
▸ $S_7 = 7/2 \cdot (2 + 20) = 7/2 \cdot 22 = \mathbf{77}$.
Apply $S_n = n/2 \cdot (a + \ell)$.

Example 2: Find 1 + 3 + 5 + … + 19 (sum of first 10 odd numbers).

▸ $a = 1$, $d = 2$, $\ell = 19$. $n = (19 - 1)/2 + 1 = \mathbf{10}$.
Find count.
▸ $S_{10} = 10/2 \cdot (1 + 19) = 5 \cdot 20 = \mathbf{100}$.
Sum of first 10 odd numbers always = 10^2.

Applied example — Weekly savings total

Naledi saves R100 in week 1, increasing R20 each week. How much has she saved after 12 weeks?

- $a = 100$, $d = 20$, $n = 12$

Arithmetic series.

- $S_{12} = 12/2[2(100) + 11(20)] = 6(420) = \text{R}2\ 520$

$S_n = n/2[2a + (n - 1)d]$.

EXERCISES (EASY → DIFFICULT)

1. Find $1 + 2 + 3 + \dots + 10$.
2. Find $2 + 4 + 6 + \dots + 20$.
3. Find $2 + 5 + 8 + \dots + 20$.
4. Find the sum of the first 20 terms of 3, 7, 11, …
5. Find the sum of the first 50 natural numbers.
6. A theatre has 20 seats in row 1 and 2 more in each next row, for 15 rows. How many seats in total?
7. Naledi saves R100 in week 1, R120 in week 2, R140 in week 3, … How much has she saved after 12 weeks?
8. How many terms of $3 + 6 + 9 + \dots$ give a sum of 165?
9. Find the sum of the first 15 terms of 5, 8, 11, …
10. The sum of the first n terms of an arithmetic series is $S_n = 2n^2 + 3n$. Find the 5th term.

12.P5 Finite geometric series

$$S = a + ar + ar^2 + ar^3 + ar^4$$

$$rS = \quad ar + ar^2 + ar^3 + ar^4 + a$$

DEFINITIONS

Geometric series	The sum of a geometric sequence: $S_n = a(r^n - 1)/(r - 1)$ for $r \neq 1$.
Partial sum (S_n)	The total of the first n terms.
Sum trick ($S - rS$)	Multiplying S by r and subtracting cancels most terms, leaving only the first and the r^n-shifted last.

KEY FORMULAS

- **Geometric series:** sum of a geometric sequence
- **$S_n = a(r^n - 1)/(r - 1)$** (when $r \neq 1$)
- **Equivalent form:** $S_n = a(1 - r^n)/(1 - r)$
- **Special case r = 1:** $S_n = a \cdot n$
- **Use the form** with positive denominator (cleaner)

WORKED EXAMPLES

Example 1: Find 3 + 6 + 12 + 24 + 48 (5 terms).

▸ $a = 3$, $r = 2$, $n = 5$.

Read off.

▸ $S_5 = 3(2^5 - 1)/(2 - 1) = 3 \cdot 31 / 1 = \mathbf{93}$.

Apply formula.

Example 2: Find 2 + 6 + 18 + 54 (4 terms).

▸ $a = 2$, $r = 3$, $n = 4$.

Read off.

▸ $S_4 = 2(3^4 - 1)/(3 - 1) = 2(80)/2 = \mathbf{80}$.

Apply formula.

Applied example — A chain message

A message is sent to 2 people, who each send to 2 more, for 6 rounds. How many messages in total?

▸ $a = 2$, $r = 2$, $n = 6$

Geometric series 2 + 4 + 8 + …

▸ $S_6 = 2(2^6 - 1)/(2 - 1) = 126$ messages

$S_n = a(r^n - 1)/(r - 1)$.

EXERCISES (EASY → DIFFICULT)

1. Find 1 + 2 + 4 + 8 + 16 + 32.
2. Find 3 + 6 + 12 + 24 (4 terms).
3. Find 5 + 15 + 45 + 135 (4 terms).
4. Find the sum of the first 6 terms of 1, 3, 9, …
5. Find the sum of the first 5 terms of 4, 8, 16, …
6. A chain message is sent to 2 people, who each send to 2 more, and so on for 6 rounds. How many messages are sent in total? (2 + 4 + … , 6 terms)
7. Grains are placed on a chessboard: 1 on the first square, doubling each square, for the first 10 squares. Find the total.
8. A bouncing ball travels 1 m, then ½ m, then ¼ m, … Find the distance over the first 4 terms.
9. How many terms of 2 + 6 + 18 + … give a sum of 728?
10. The first three terms of a geometric series sum to 21, with first term 3 ($r > 0$). Find r.

12.P6 Infinite geometric series

$4 + 2 + 1 + 0.5 + 0.25 + \ldots \to 8$

$|r| < 1$: each term shrinks; sum approaches a finite limit.

DEFINITIONS

Infinite series	A series that adds infinitely many terms.
Convergence	When an infinite series adds up to a finite number. For a geometric series, this happens iff $|r| < 1$.
Divergence	When an infinite series does NOT add up to a finite number — the partial sums grow without limit.
Sum to infinity (S_∞)	The limit $a/(1 - r)$ — only meaningful when $|r| < 1$.

KEY FORMULAS

- **Convergence condition:** $|r| < 1$ (must be strictly less than 1 in absolute value)
- **Sum to infinity:** $S_\infty = a / (1 - r)$ (only when $|r| < 1$)
- **If $|r| \geq 1$:** the series DIVERGES (no finite sum)
- **Why:** as $n \to \infty$, $r^n \to 0$ (only when $|r| < 1$)
- **Common cases:** $r = 1/2, 1/3, 0.1, 0.5$

WORKED EXAMPLES

Example 1: Find 4 + 2 + 1 + 0.5 + …

▸ **Check convergence:** $r = 2/4 = 1/2$. $|1/2| < 1$ — series converges.
Always check first.

▸ $S_\infty = 4 / (1 - 1/2) = 4 / 0.5 = \mathbf{8}$.
Apply formula.

Example 2: Find 9 + 3 + 1 + 1/3 + …

▸ $a = 9$, $r = 3/9 = 1/3$. $|1/3| < 1$.
Convergence check.

▸ $S_\infty = 9 / (1 - 1/3) = 9 / (2/3) = 9 \times 3/2 = \mathbf{13.5}$.
Divide.

Example 3: Does 2 + 4 + 8 + 16 + … converge?

▸ $r = 4/2 = 2$. $\mathbf{|2| \geq 1}$.
Outside convergence range.

▸ **Diverges** — no sum to infinity.
Each term gets bigger; sum grows without bound.

Applied example — Total swing of a pendulum

A pendulum swings 40 cm, then 0,8 of each previous swing. Find the total distance it swings.

- $a = 40$, $r = 0.8$, $|r| < 1$
  Converges.
- $S_\infty = 40/(1 - 0.8) = 200$ cm
  $S_\infty = a/(1 - r)$.

EXERCISES (EASY → DIFFICULT)

1. State the condition for an infinite geometric series to converge.
2. Find S_∞ for $4 + 2 + 1 + \dots$
3. Find S_∞ for $9 + 3 + 1 + \dots$
4. Find S_∞ for $6 + 3 + 1.5 + \dots$
5. Does $2 + 4 + 8 + \dots$ converge?
6. A ball bounces to ¾ of its height each time, first drop 2 m. Find the total vertical distance it falls ($2 + 2\cdot\frac{3}{4} + \dots$).
7. A pendulum swings 40 cm, then 0.8 of each previous swing. Find the total distance it swings.
8. Write the recurring decimal 0,7777… as a fraction using S_∞.
9. A geometric series has first term 8 and sum to infinity 16. Find r.
10. For which x does $1 + x + x^2 + \dots$ converge, and to what?

Past Paper — Number Patterns (quadratic sequence)

Past-paper number patterns — quadratic sequences

DEFINITIONS

Quadratic sequence	A sequence whose second differences are constant; its nth term has the form a n squared plus b n plus c.
First difference	The difference between two consecutive terms of the sequence.
Second difference	The difference between consecutive first differences; it is constant for a quadratic sequence.

KEY FORMULAS

- **Quadratic sequence:** $T_n = an^2 + bn + c$
- **Second difference = 2a**
- **First first-difference ($T_2 - T_1$) = 3a + b**
- **First term: $T_1 = a + b + c$**
- **Arithmetic nth term:** $T_n = a + (n - 1)d$

WORKED EXAMPLES

Example 1: Quadratic sequence: 3 ; 8 ; 17 ; 30 ; … Write down the next TWO terms. *(2)*

- First differences: 5 ; 9 ; 13 ; …

Subtract consecutive terms. They are not constant, so the sequence is not arithmetic.

- Second difference: 4 (constant)

The differences of the first differences are constant — this confirms a quadratic sequence.

- Next first differences: 17 ; 21 (add 4 each time)

Continue the first differences by adding the constant second difference.

- **Next two terms: 47 ; 68**

30 + 17 = 47, then 47 + 21 = 68.

Example 2: Determine the nth term T_n of the sequence 3 ; 8 ; 17 ; 30 ; … *(4)*

- $2a = 4 \rightarrow a = 2$

For $T_n = an^2 + bn + c$, the constant second difference equals 2a.

- $3a + b = 5 \rightarrow 6 + b = 5 \rightarrow b = -1$

The first of the first differences ($T_2 - T_1$) equals 3a + b.

- $a + b + c = 3 \rightarrow 2 - 1 + c = 3 \rightarrow c = 2$

The first term T_1 equals a + b + c.

- $\mathbf{T_n = 2n^2 - n + 2}$

Substitute a, b and c. Check: $T_4 = 32 - 4 + 2 = 30$.

Example 3: Which term of the sequence is equal to 782? *(3)*

- $2n^2 - n + 2 = 782$

Set the nth term equal to 782.

- $2n^2 - n - 780 = 0$

Write the equation in standard form.

- $(n - 20)(2n + 39) = 0$

Factorise the quadratic.

- $n = 20$ (reject $n = -39/2$)

n must be a positive whole number, so reject the negative fraction.

- **The 20th term equals 782.**

Check: 2(400) − 20 + 2 = 782 .

Example 4: Determine the nth term of the FIRST differences of the sequence. *(2)*

- First differences: 5 ; 9 ; 13 ; 17 ; …

These themselves form an arithmetic sequence.

- $a = 5, \quad d = 4$

First term 5, common difference 4.

- $\mathbf{D_n = 4n + 1}$

$D_n = a + (n - 1)d = 5 + 4(n - 1) = 4n + 1$.

Applied example — Stones in a paving pattern

Rings of paving use 3, 8, 17, 30, … stones. Which ring uses 71 stones?

- $T_n = 2n^2 - n + 2$

Quadratic sequence (second diff 4).

- $2n^2 - n + 2 = 71 \rightarrow 2n^2 - n - 69 = 0$

Set equal to 71.

- $n = 6$

Solve; reject the negative root.

EXERCISES (EASY → DIFFICULT)

1. Write the first three terms of the quadratic sequence $T_n = n^2 + 2n$.
2. Find the second difference of 3, 8, 17, 30, …
3. For 3, 8, 17, 30, … find a ($T_n = an^2 + bn + c$).
4. Find the nth term of 3, 8, 17, 30, …
5. Hence find T_{10} of 3, 8, 17, 30, …
6. A pattern of paving stones grows as 3, 8, 17, 30, … per ring. Which ring uses 71 stones?
7. For 2, 6, 12, 20, … find T_n.
8. Find the first-difference formula for $T_n = 2n^2 - n + 2$.
9. A quadratic sequence has $T_1 = 4$ and first differences 3, 7, 11, … Find T_4.
10. In 3, 8, 17, 30, … find the value of the first term whose first difference exceeds 40.

End of Chapter 8 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Find the next term of 3, 7, 11, 15, …
2. Find a (where $T_n = an^2 + bn + c$) for 1, 4, 9, 16.
3. For 1, 5, 25, … find T_5.
4. Find the sum of the first 20 terms of 3, 7, 11, …
5. Find the sum of the first 5 terms of 4, 8, 16, …
6. A ball bounces to ¾ of its height each time, first drop 2 m. Find the total vertical distance it falls ($2 + 2\cdot\frac{3}{4} + \ldots$).
7. For 2, 6, 12, 20, … find T_n.
8. Lerato's salary starts at R8 000 and rises by R500 each year. What is her salary in year 6?
9. For 3, 8, 17, 30, … find T_5.
10. Three numbers x, x + 3, x + 9 form a geometric sequence. Find x.

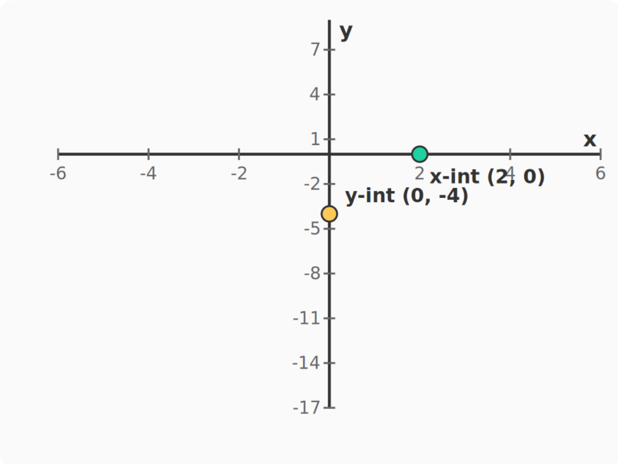
CHAPTER 9

Functions

12.F1 Straight Line function (with inverse)
12.F2 Quadratic function (with restricted-domain inverse)
12.F3 Hyperbolic function (self-inverse)
12.F4 Exponential function (with logarithm inverse)

12.F1 Straight Line function

Straight Line function (with inverse)



DEFINITIONS

Linear function	A function of the form $y = mx + c$. Its graph is always a straight line.
Gradient	The slope of the line: how steep it is. $m = (y_2 - y_1)/(x_2 - x_1)$.
Y-intercept	The y-value where the line crosses the y-axis. Found by setting $x = 0$.
Inverse function	A function that "undoes" the original. For $y = mx + c$, swap x and y to find it.

KEY FORMULAS

- **Standard form:** $y = mx + c$
- **To find the inverse:** swap x and y, then solve for y
- **Inverse of $y = mx + c$:** $y = (x - c) / m$
- **Reflection rule:** a function and its inverse reflect across $y = x$
- **Domain & range** swap between $f(x)$ and $f^{-1}(x)$

WORKED EXAMPLES

Example 1: Find the inverse of $y = 2x - 4$.

▸ Swap: $x = 2y - 4$.

Step 1.

▸ Add 4: $x + 4 = 2y$. Divide by 2: $\mathbf{y = (x + 4)/2}$.

Step 2 — solve for y.

Example 2: Find the inverse of $y = 3x + 6$.

▸ Swap: $x = 3y + 6$.

First step always.

▸ Subtract 6, divide by 3: $\mathbf{y = (x - 6)/3}$.

Inverse function.

Example 3: A line passes through (1, 3) and (4, 9). Find its inverse.

▸ Gradient: $m = (9 - 3)/(4 - 1) = \mathbf{2}$. Point-slope: $y - 3 = 2(x - 1) \rightarrow y = 2x + 1$.

Get the original equation first.

▸ Swap: $x = 2y + 1$. Solve: $\mathbf{y = (x - 1)/2}$.

Inverse.

Applied example — Distance from a taxi fare

A taxi fare is $y = 12 + 8x$ rand for x km. Find the inverse and the distance for a R60 fare.

▸ $x = (y - 12)/8$

Swap and solve for x.

▸ $x = (60 - 12)/8 = 6$ km

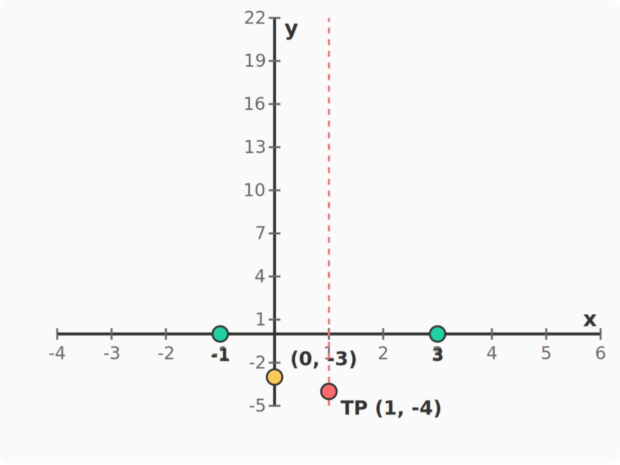
Substitute $y = 60$.

EXERCISES (EASY → DIFFICULT)

1. Find the gradient of $y = 2x - 4$.
2. Find the y-intercept of $y = 3x - 9$.
3. Find the x-intercept of $y = 3x - 9$.
4. Find the inverse of $y = 2x - 4$.
5. Find the inverse of $y = -2x + 6$.
6. A plumber charges $y = 350 + 250x$ rand for x hours. Express x in terms of y (the inverse) to find hours from the bill.
7. Rands convert to dollars by $d = x/18$ (x in rand). Find the inverse to convert dollars back to rand.
8. A line and its inverse are reflections in which line?
9. Find where $y = 2x - 4$ meets its inverse.
10. A taxi fare is $y = 12 + 8x$ rand for x km. Write the inverse and use it to find the distance for a R60 fare.

12.F2 Quadratic function

Quadratic function (with restricted-domain inverse)



DEFINITIONS

Quadratic function	A function of the form $y = ax^2 + bx + c$. Its graph is a parabola.
Parabola	The U-shaped (or ∩-shaped) curve made by a quadratic.
Turning point	The lowest (or highest) point of the parabola. Has coordinates (p, q) in turning-point form.
Discriminant	$b^2 - 4ac$. Tells you how many real roots: positive=2, zero=1, negative=0.
Axis of symmetry	The vertical line $x = p$ that splits the parabola into two mirror halves.

KEY FORMULAS

- **Standard form:** $y = ax^2$
- **Inverse rule:** $y = ax^2$ with $x \geq 0$ has inverse $y = \sqrt{x/a}$
- **Restriction needed:** the parabola fails the horizontal line test
- **Reflection:** the inverse is the parabola reflected across $y = x$
- **Domain of f:** $x \geq 0$ **Range of f:** $y \geq 0$
- **Domain of f^{-1}:** $x \geq 0$ **Range of f^{-1}:** $y \geq 0$

WORKED EXAMPLES

Example 1: Find the inverse of $y = x^2$ for $x \geq 0$.

▸ Swap: $x = y^2$.

Standard inverse step.

▸ Take square root: $\mathbf{y = \sqrt{x}}$.

Take positive root since $y \geq 0$.

Example 2: Find the inverse of $y = 2x^2$ for $x \geq 0$.

- Swap: $x = 2y^2$.

First step.

- Divide by 2: $y^2 = x/2$. Take √: $\mathbf{y = \sqrt{x/2}}$.

Solve for y.

Example 3: Why does $y = x^2$ (no restriction) have no inverse function?

- Pick $y = 4$. We need a single x. But $x = 2$ AND $x = -2$ both give $y = 4$.

Two answers, not one.

- An inverse must be a function — each input gives one output. So we need to pick one branch only ($x \geq 0$ or $x \leq 0$).

Domain restriction fixes the ambiguity.

Applied example — Time to reach a height

A ball's height is $h = 5t^2$ ($t \geq 0$). Write t in terms of h and find t when $h = 20$ m.

- $t = \sqrt{h/5}$

Inverse for $t \geq 0$.

- $t = \sqrt{20/5} = \sqrt{4} = 2$ s

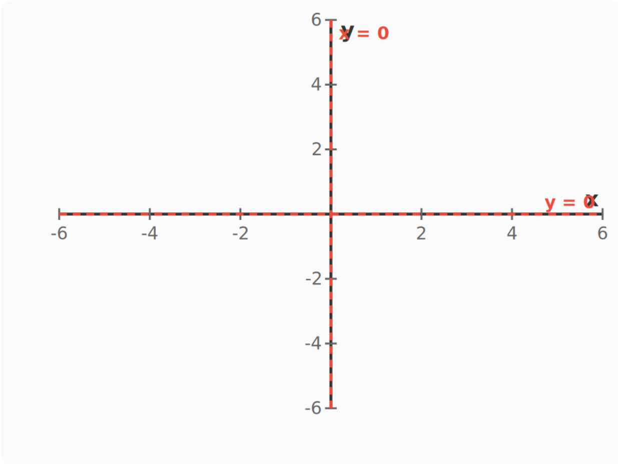
Substitute $h = 20$.

EXERCISES (EASY → DIFFICULT)

1. Why must the domain of $y = x^2$ be restricted to obtain an inverse function?
2. Find the inverse of $y = x^2$ for $x \geq 0$.
3. Find the inverse of $y = x^2$ for $x \leq 0$.
4. State the range of $y = x^2$, $x \geq 0$.
5. Reflect the point (3, 9) on $y = x^2$ in the line $y = x$.
6. A ball's height is $h = 5t^2$ ($t \geq 0$ seconds). Write t in terms of h to find the time to reach height h.
7. The area of a circle is $A = \pi r^2$. Express r in terms of A (the inverse).
8. Find the inverse of $y = 2x^2$, $x \geq 0$.
9. Is the inverse of $y = x^2$ ($x \in \mathbb{R}$) a function? Explain.
10. Find the inverse of $y = (x - 1)^2$, $x \geq 1$.

12.F3 Hyperbolic function

Hyperbolic function (self-inverse)



DEFINITIONS

Hyperbola	The curve made by $y = a/(x - p) + q$. It has two separate branches.
Asymptote	A line that the curve approaches but never touches. Hyperbolas have two: $x = p$ and $y = q$.
Self-inverse	A function that is its own inverse — for $y = a/x$, swapping x and y gives the same equation.

KEY FORMULAS

- **Standard form:** $y = a/x$
- **Inverse:** swap x, y → $x = a/y$ → $y = a/x$ *(same equation!)*
- **Self-inverse:** the graph is its own reflection across $y = x$
- **Asymptotes:** $x = 0$ and $y = 0$ — they are themselves symmetric across $y = x$
- **Domain:** $x \neq 0$ **Range:** $y \neq 0$

WORKED EXAMPLES

Example 1: Show that y = 2/x is its own inverse.

▸ Swap x, y: $x = 2/y$. Multiply by y: $xy = 2$.

Standard inverse procedure.

▸ Divide by x: $\mathbf{y = 2/x}$. Same equation as the original!

Self-inverse confirmed.

Example 2: Pick the point (2, 3) on y = 6/x. Where is the corresponding point on the inverse?

▸ Inverse: swap coordinates of every point.

(x, y) becomes (y, x).

▸ (2, 3) reflects to **(3, 2)**.

And 6/3 = 2 — the inverse passes through (3, 2).

Example 3: For y = −4/x, what are the asymptotes of its inverse?

- ▸ Self-inverse, so the inverse is also $y = -4/x$.
  Same equation.
- ▸ Asymptotes: **x = 0 and y = 0** — same as the original.
  Both asymptotes lie on the y = x reflection axis pair.

Applied example — Workers and time

A job takes $y = 240/x$ hours for x workers. How many workers finish it in 10 hours?

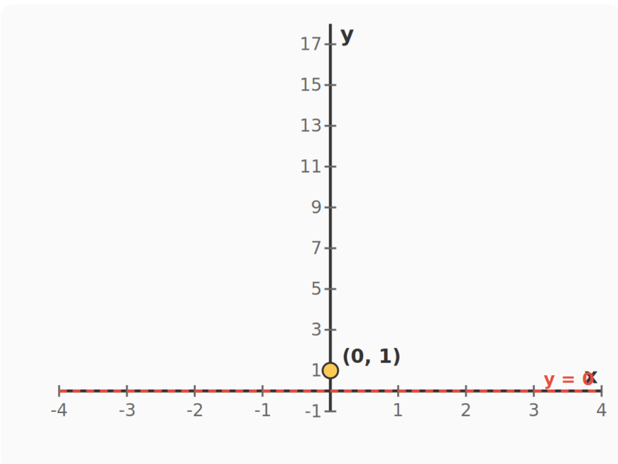
- ▸ $10 = 240/x$
  Substitute y = 10.
- ▸ $x = 24$ workers
  Solve the hyperbola equation.

EXERCISES (EASY → DIFFICULT)

1. State the asymptotes of $y = 1/x$.
2. Find the inverse of $y = 4/x$.
3. Why is $y = k/x$ called self-inverse?
4. State the asymptotes of $y = 2/(x - 3)$.
5. State the asymptotes of $y = 1/x + 2$.
6. A job takes $y = 240/x$ hours when x workers share it. Sketch its shape and state the asymptotes.
7. Using $y = 240/x$ hours, how long do 8 workers take, and how many workers finish it in 10 hours?
8. Find the inverse of $y = 2/x$.
9. Find where $y = 4/x$ meets $y = x$.
10. Find the inverse of $y = 3/x + 1$ (express y).

12.F4 Exponential function

Exponential function (with logarithm inverse)



DEFINITIONS

Exponential function	A function of the form $y = a \cdot b^x + q$ where $b > 0$ and $b \neq 1$.
Logarithm	The inverse of the exponential. log_b x asks: "what power of b gives x?"
Horizontal asymptote	A horizontal line that the exponential curve approaches but never crosses. For $y = a \cdot b^x + q$, it is $y = q$.
Growth vs decay	When $b > 1$ the function grows; when $0 < b < 1$ the function decays.

KEY FORMULAS

- **Standard form:** $y = b^x$
- **Inverse:** $y = \log_b(x)$ — the logarithm
- **Definition:** $\log_b(x) = y$ means $b^y = x$
- **Identity:** $\log_b(b^x) = x$ and $b^{\log_b(x)} = x$
- **Reflection:** the log graph is the exponential reflected across $y = x$
- **Domain of log:** $x > 0$ **Range of log:** $y \in \mathbb{R}$

WORKED EXAMPLES

Example 1: Find the inverse of $y = 2^x$.

▸ Swap: $x = 2^y$.

Inverse step 1.

▸ Take log base 2 of both sides: $\mathbf{y = \log_2(x)}$.

Logarithm undoes the exponent.

Example 2: Solve $3^x = 81$.

▸ Take $\log_3$ of both sides: $x = \log_3(81)$.

Apply the inverse.

▸ Since $3^4 = 81$: $\mathbf{x = 4}$.

Recognise 81 as a power of 3.

Example 3: Compare the y-intercepts of $y = 2^x$ and $y = \log_2(x)$.

▸ $\mathbf{y = 2^x}$**:** y-intercept is (0, 1) — passes through 1 on the y-axis.

$2^0 = 1$.

▸ $\mathbf{y = \log_2(x)}$**:** $\log_2(0)$ is undefined. Curve has a vertical asymptote at $x = 0$ — no y-intercept.

Inverse swaps x and y, so the (0, 1) y-intercept becomes the (1, 0) x-intercept on the log graph.

Applied example — Time for a colony to grow

A colony grows as $N = 2^t$ (t in hours). After how long does it reach 1 000?

▸ $t = \log_2 N$

Inverse of the exponential.

▸ $t = \log 1000 / \log 2 \approx 9.97$ hours

Change of base, then evaluate.

EXERCISES (EASY → DIFFICULT)

1. Evaluate $\log_2 8$.
2. Evaluate $\log_{10} 1000$.
3. Find the inverse of $y = 2^x$.
4. Solve $3^x = 27$ using logs.
5. State the asymptote of $y = 2^x$.
6. A colony of bacteria is $N = 2^t$ (t in hours). Write the inverse to find the time to reach a population N.
7. Using $N = 2^t$, after how long does the colony reach 1 000? (2 dp)
8. An investment grows as $A = 5^x$ (x in years). Solve $5^x = 20$ to find when it is 20× the start (2 dp).
9. Find the inverse of $y = 3 \cdot 2^x$ (express y).
10. For $y = 2^x$, give where it cuts the y-axis, and where its inverse cuts the x-axis.

End of Chapter 9 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

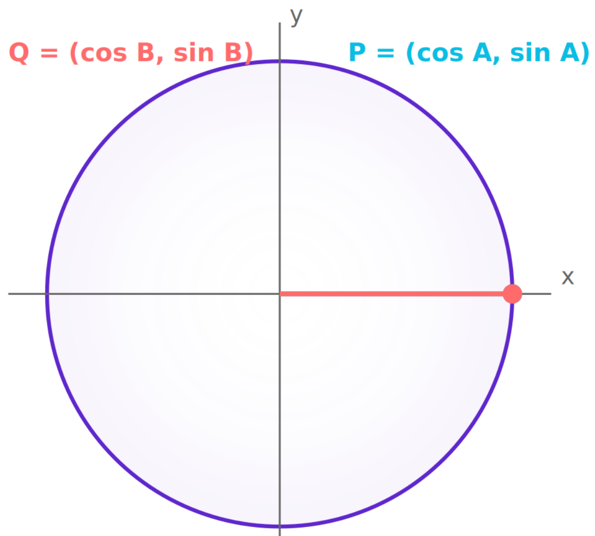
1. Find the gradient of $y = 2x - 4$.
2. Find the inverse of $y = x^2$ for $x \geq 0$.
3. Why is $y = k/x$ called self-inverse?
4. Solve $3^x = 27$ using logs.
5. Find the inverse of $y = -2x + 6$.
6. A ball's height is $h = 5t^2$ ($t \geq 0$ seconds). Write t in terms of h to find the time to reach height h.
7. Using $y = 240/x$ hours, how long do 8 workers take, and how many workers finish it in 10 hours?
8. An investment grows as $A = 5^x$ (x in years). Solve $5^x = 20$ to find when it is 20× the start (2 dp).
9. Find where $y = 2x - 4$ meets its inverse.
10. Find the inverse of $y = (x - 1)^2$, $x \geq 1$.

CHAPTER 12

Trigonometry

12.1 Compound angle identities
12.2 Double angle identities
12.3 Solving equations with identities
12.4 Applications of trigonometric functions

12.1 Compound angle identities



rotate. The chord PQ depends only on the angle $(A - B)$

DEFINITIONS

Compound angle	An angle formed by adding or subtracting two angles, like $A + B$ or $A - B$.
Identity	An equation that is true for every value of the variable, not just some.
Unit circle	A circle with radius 1 centred at the origin. Every point on it has coordinates $(\cos \theta, \sin \theta)$.

KEY FORMULAS

- **$\cos(A - B)$** = $\cos A \cos B + \sin A \sin B$
- **$\cos(A + B)$** = $\cos A \cos B - \sin A \sin B$
- **$\sin(A + B)$** = $\sin A \cos B + \cos A \sin B$
- **$\sin(A - B)$** = $\sin A \cos B - \cos A \sin B$
- *Memory trick:* for cos, the middle sign FLIPS. For sin, the middle sign is the SAME as the original.

WORKED EXAMPLES

Example 1: Find the EXACT value of cos 75°.

▸ Write 75° as a sum of special angles: **$75° = 45° + 30°$**.
We know exact values for 30°, 45°, 60°. Pick a sum that hits 75°.
▸ Apply the formula: **$\cos(45° + 30°) = \cos 45° \cos 30° - \sin 45° \sin 30°$**.
Cosine of a SUM — middle sign FLIPS to minus.
▸ Substitute: $= (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(1/2)$.
Use the special-angle values you memorised in Grade 10.
▸ Simplify: $= \sqrt{6}/4 - \sqrt{2}/4$.
Multiply each pair.
▸ **$\cos 75° = (\sqrt{6} - \sqrt{2}) / 4$**.
Combine over a common denominator.

Example 2: Find the EXACT value of sin 15° using sin(45° − 30°).

▸ Apply: **$\sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$**.
Sine of a DIFFERENCE — middle sign STAYS as minus.
▸ Substitute: $= (\sqrt{2}/2)(\sqrt{3}/2) - (\sqrt{2}/2)(1/2)$.
Use special-angle values.
▸ Simplify: $= \sqrt{6}/4 - \sqrt{2}/4$.
Multiply each pair.
▸ **$\sin 15° = (\sqrt{6} - \sqrt{2}) / 4$**.
Same exact value as cos 75° — that's because sin 15° = cos 75° (co-function).

Example 3: Simplify cos 80° cos 20° + sin 80° sin 20°.

▸ Spot the pattern: $\cos A \cos B + \sin A \sin B$ is **$\cos(A - B)$**.
Match the formula in REVERSE — recognise the structure first.
▸ So this equals **$\cos(80° - 20°) = \cos 60°$**.
Substitute the angles.
▸ **$= 1/2$**.
Use the special-angle value.

Applied example — Exact cosine of a combined angle

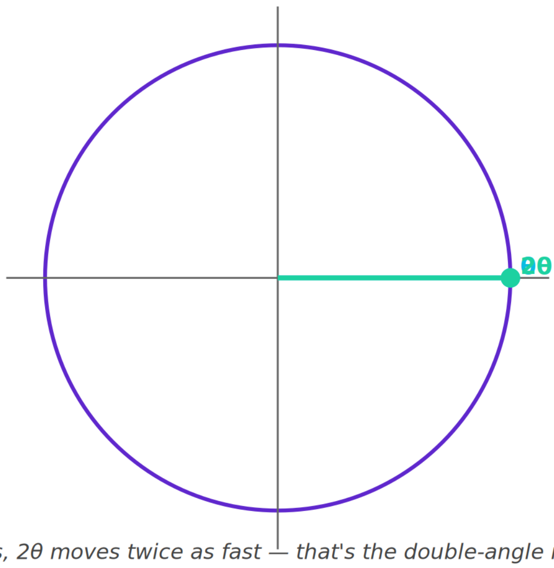
Two ramps make 45° and 30° with the ground. Find the exact value of cos 75°.

▸ $\cos 75° = \cos(45° + 30°)$
Write as a compound angle.
▸ $= \cos 45° \cos 30° - \sin 45° \sin 30°$
Expand.
▸ $= (\sqrt{6} - \sqrt{2})/4$
Substitute exact values.

EXERCISES (EASY → DIFFICULT)

1. Write the expansion of sin(A + B).
2. Write the expansion of cos(A + B).
3. Write the expansion of cos(A − B).
4. Simplify sin60°cos30° + cos60°sin30°.
5. Simplify cos50°cos10° − sin50°sin10°.
6. Find the exact value of cos15° using cos(45° − 30°).
7. Find the exact value of sin75° using sin(45° + 30°).
8. Two ramps meet so their angles to the ground are 45° and 30°. Find the exact cosine of the total angle 75°.
9. Simplify cos(90° − θ).
10. Prove cos(A + B) + cos(A − B) = 2cosA cosB.

12.2 Double angle identities



ιtes, 2θ moves twice as fast — that's the double-angle rela

DEFINITIONS

Double angle	The angle 2θ — twice some angle θ.
Double-angle identity	A formula that rewrites sin 2θ or cos 2θ in terms of sin θ and cos θ.
Pythagorean identity	The relationship $\sin^2\theta + \cos^2\theta = 1$ — used to switch between sin and cos.

KEY FORMULAS

- **sin 2θ** = 2 sin θ cos θ
- **cos 2θ** = $\cos^2\theta - \sin^2\theta$
- **cos 2θ** = $1 - 2\sin^2\theta$ *(when only sin θ is known)*
- **cos 2θ** = $2\cos^2\theta - 1$ *(when only cos θ is known)*
- **tan 2θ** = $(2\tan\theta) / (1 - \tan^2\theta)$

WORKED EXAMPLES

Example 1: If sin θ = 3/5 and cos θ = 4/5, find sin 2θ and cos 2θ.

▸ For sin 2θ, use: **sin 2θ = 2 sin θ cos θ**.

Always the same formula for sin 2θ.

▸ = 2 × (3/5) × (4/5) = **24/25**.

Just multiply through.

▸ For cos 2θ, any of the three forms works. Try $\mathbf{\cos^2\theta - \sin^2\theta}$.

Both sin and cos are known, so use either form.

▸ = $(4/5)^2 - (3/5)^2$ = 16/25 − 9/25 = **7/25**.

Square each, subtract.

Example 2: If cos θ = 3/5 and θ is acute, find cos 2θ using ONLY cos θ.

▸ Choose the form that needs ONLY cos θ: $\mathbf{\cos 2\theta = 2\cos^2\theta - 1}$.

We don't need to find sin θ this way.

▸ = $2(3/5)^2 - 1$.

Substitute the known value.

▸ = 2(9/25) − 1 = 18/25 − 25/25.

Square first, then subtract.

▸ **cos 2θ = −7/25**.

Final answer — and notice it's negative because 2θ has crossed into Q2.

Example 3: Simplify 2 sin 75° cos 75°.

▸ Spot the pattern: 2 sin θ cos θ is **sin 2θ**.

Match the formula in reverse.

▸ So this equals **sin(2 × 75°) = sin 150°**.

Double the angle inside.

▸ Use reduction: sin 150° = sin(180° − 30°) = sin 30°.

Reduction formula from Grade 11.

▸ **= 1/2**.

Special-angle value.

Applied example — Doubling a spoke angle

A wheel spoke makes angle θ with sinθ = 8/17 (θ acute). Find sin2θ.

▸ cosθ = 15/17

From $\sin^2\theta + \cos^2\theta = 1$.

▸ sin2θ = 2 sinθ cosθ = 2(8/17)(15/17)

Double-angle identity.

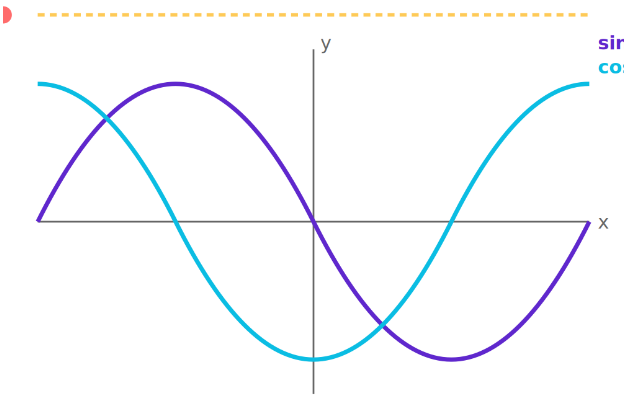
▸ = 240/289

Evaluate.

EXERCISES (EASY → DIFFICULT)

1. Write sin2θ in terms of sinθ and cosθ.
2. Write the three forms of cos2θ.
3. If sinθ = 3/5 and cosθ = 4/5, find sin2θ.
4. Using the same θ, find cos2θ.
5. Simplify 2sin15°cos15°.
6. A wheel's spoke makes angle θ with sinθ = 8/17 (θ acute). Find sin2θ for the doubled angle.
7. For the same spoke, find cos2θ.
8. Simplify $2\cos^2 35° - 1$.
9. Express sin2θ/(1 + cos2θ) more simply.
10. If cosθ = 12/13 (θ acute), find cos2θ.

12.3 Solving equations with identities



Trig equations: find x where the wave meets the horizontal line.

DEFINITIONS

Trigonometric equation	An equation that contains a trig function (sin, cos, tan) of an unknown angle.
Reference angle	The acute angle between the terminal arm and the x-axis. Used to find ALL solutions.
General solution	A formula that lists every angle that solves the equation, usually using +k·360° or +k·180°.
CAST diagram	A four-quadrant chart showing which trig ratios are positive: All, Sin, Tan, Cos.

KEY FORMULAS

- **Strategy:** Reduce the equation to ONE trig function in ONE angle.
- If the equation has **2θ AND θ**: use a double-angle identity.
- If the equation has **(A ± B)**: use a compound-angle identity.
- Then **factor** (often common factor or quadratic) and solve each piece.
- Always **check the range** and write down ALL solutions in it.

WORKED EXAMPLES

Example 1: Solve $\sin 2\theta = \sin \theta$ for $0° \le \theta \le 360°$.

- ▸ Spot 2θ AND θ — replace $\sin 2\theta$ with $\mathbf{2 \sin \theta \cos \theta}$.
  Use the double-angle identity to reduce to one angle.
- ▸ Equation becomes: $\mathbf{2 \sin \theta \cos \theta = \sin \theta}$.
  After substituting.
- ▸ Move everything to one side: $\mathbf{2 \sin \theta \cos \theta - \sin \theta = 0}$.
  Standard zero form. Don't divide by $\sin \theta$ — you'd lose solutions.
- ▸ Factor out $\sin \theta$: $\mathbf{\sin \theta\,(2 \cos \theta - 1) = 0}$.
  $\sin \theta$ is the common factor.
- ▸ Solve each factor. $\mathbf{\sin \theta = 0}$ gives $\theta = 0°, 180°, 360°$.
  Where sine crosses zero in the range.
- ▸ $\mathbf{2 \cos \theta - 1 = 0} \to \cos \theta = 1/2 \to \theta = 60°$ (Q1) or $300°$ (Q4).
  Cosine is positive in Q1 and Q4.
- ▸ All solutions: $\mathbf{\theta = 0°, 60°, 180°, 300°, 360°}$.
  Five solutions in the range.

Example 2: Solve $\cos 2\theta + \cos \theta = 0$ for $0° \le \theta \le 360°$.

- ▸ There's 2θ AND θ — pick a $\cos 2\theta$ form. Use $\mathbf{\cos 2\theta = 2 \cos^2\theta - 1}$.
  We chose this form because the equation already has $\cos \theta$ — keeping everything in cos avoids introducing sin.
- ▸ Substitute: $\mathbf{2 \cos^2\theta - 1 + \cos \theta = 0}$.
  Now the equation is in $\cos \theta$ only.
- ▸ Rearrange: $\mathbf{2 \cos^2\theta + \cos \theta - 1 = 0}$.
  Standard quadratic form in $\cos \theta$.
- ▸ Factor: $\mathbf{(2 \cos \theta - 1)(\cos \theta + 1) = 0}$.
  Factorise like any quadratic.
- ▸ $\mathbf{\cos \theta = 1/2} \to \theta = 60°$ or $300°$.
  First factor.
- ▸ $\mathbf{\cos \theta = -1} \to \theta = 180°$.
  Second factor.
- ▸ All solutions: $\mathbf{\theta = 60°, 180°, 300°}$.
  Three solutions in the range.

Example 3: Solve $\sin(\theta + 30°) = \cos \theta$ for $0° \le \theta \le 360°$.

- ▸ There's $(\theta + 30°)$ — expand using $\mathbf{\sin(A + B) = \sin A \cos B + \cos A \sin B}$.
  Compound angle to break it apart.
- ▸ $\mathbf{\sin \theta \cos 30° + \cos \theta \sin 30° = \cos \theta}$.
  Apply the formula.
- ▸ Substitute: $\mathbf{(\sqrt{3}/2) \sin \theta + (1/2) \cos \theta = \cos \theta}$.
  Use special-angle exact values.
- ▸ Move all $\cos \theta$ terms to one side: $\mathbf{(\sqrt{3}/2) \sin \theta = (1/2) \cos \theta}$.
  Subtract $(1/2) \cos \theta$ from both sides.
- ▸ Divide both sides by $\cos \theta$: $\mathbf{\tan \theta = 1/\sqrt{3}}$.
  sin/cos = tan. Be sure $\cos \theta \ne 0$; check those cases separately if needed.
- ▸ $\mathbf{\theta = 30°}$ (Q1) **or** $\mathbf{210°}$ (Q3).
  tan is positive in Q1 and Q3.

Applied example — When is the wave at half height?

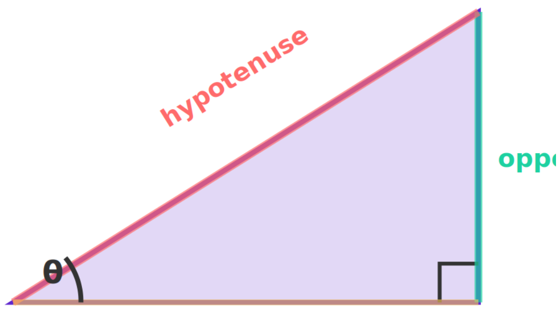
A wave is $h = \sin\theta$. For $0° \le \theta \le 360°$, find the angles where $h = ½$.

- ▸ $\sin\theta = ½$
  Set the height to ½.
- ▸ $\theta = 30°$ or $150°$
  Sine is positive in quadrants I and II.

EXERCISES (EASY → DIFFICULT)

1. Solve $\sin\theta = ½$ for $0° \le \theta \le 360°$.
2. Solve $\cos\theta = 0$ for $0° \le \theta \le 360°$.
3. Solve $\tan\theta = 1$ for $0° \le \theta \le 360°$.
4. Solve $2\cos\theta = \sqrt{3}$ for $0° \le \theta \le 360°$.
5. Solve $\sin 2\theta = \sqrt{3}/2$ for $0° \le \theta \le 90°$.
6. A wave is modelled by $h = \sin\theta$. For $0° \le \theta \le 360°$, at which angles is the height ½?
7. Solve $\sin 2\theta = \sin\theta$ for $0° \le \theta \le 360°$.
8. Solve $\cos 2\theta = \cos\theta$ for $0° \le \theta \le 360°$.
9. Solve $2\sin^2\theta = \sin\theta$ for $0° \le \theta \le 360°$.
10. Solve $\cos 2\theta + 3\cos\theta = 1$ for $0° \le \theta \le 360°$.

12.4 Applications of trigonometric functions



sin θ = opposite ÷ hypotenuse

DEFINITIONS

Angle of elevation	The angle measured upward from the horizontal — looking up at something tall.
Angle of depression	The angle measured downward from the horizontal — looking down at something below.
Bearing	A direction measured clockwise from north, given as a 3-digit number like 045° or 270°.
Sine rule	$a/\sin A = b/\sin B = c/\sin C$ — links sides and opposite angles in any triangle.
Cosine rule	$a^2 = b^2 + c^2 - 2bc\cdot\cos A$ — generalises Pythagoras to any triangle.

KEY FORMULAS

- **SOH CAH TOA** for right triangles:
- **sin θ** = opposite ÷ hypotenuse
- **cos θ** = adjacent ÷ hypotenuse
- **tan θ** = opposite ÷ adjacent
- **Sine rule** (any triangle): $a/\sin A = b/\sin B = c/\sin C$
- **Cosine rule** (any triangle): $a^2 = b^2 + c^2 - 2bc \cos A$
- **Area rule**: Area = ½ ab sin C

WORKED EXAMPLES

Example 1 — Ladder. A 10 m ladder leans at 70° to the ground. How high up the wall does it reach?

▸ Draw and label: hypotenuse = ladder = 10 m, angle to ground = 70°.
Ladder is the slanted side, so it's the hypotenuse.
▸ Wall height (opposite) is unknown — call it h.
h is across from the 70° angle.
▸ Choose the ratio that uses opposite and hypotenuse: **sin 70° = h ÷ 10**.
SOH — sine = opposite ÷ hypotenuse.
▸ Rearrange: **h = 10 × sin 70°**.
Multiply both sides by 10.
▸ **h ≈ 9.40 m**.
Calculator in DEG mode.

Example 2 — Lighthouse. A boat sees the top of a lighthouse at an elevation angle of 30° from 150 m away on the sea. Find the lighthouse height.

▸ Draw and label: adjacent (sea distance) = 150 m, angle of elevation = 30°.
Elevation angle is measured upwards from horizontal.
▸ Lighthouse height = opposite = h.
Vertical side, across from the 30° angle.
▸ Choose ratio for opposite and adjacent: **tan 30° = h ÷ 150**.
TOA — tangent = opposite ÷ adjacent.
▸ Rearrange: **h = 150 × tan 30° = 150 × (1/√3)**.
Use the special-angle value.
▸ **h ≈ 86.60 m**.
Final answer.

Example 3 — Two buildings. From the top of a 50 m tall tower, the angle of depression to a car on the ground is 35°. How far is the car from the tower's base?

▸ Angle of depression = angle of elevation from the car (alternate angles).
So the angle at the car looking up to the tower top is also 35°.
▸ From the car: opposite (tower) = 50 m, angle = 35°, adjacent (distance) = d.
Same right triangle, viewed from below.
▸ Choose: **tan 35° = 50 ÷ d**.
TOA again — opposite over adjacent.
▸ Rearrange: **d = 50 ÷ tan 35°**.
Multiply both sides by d, then divide by tan 35°.
▸ **d ≈ 71.41 m**.
Calculator finishes the job.

Applied example — Height of a tower

A surveyor stands 60 m from a tower and measures the top at 35° elevation. Find the tower's height.

- $\tan 35° = h/60$
  Opposite over adjacent.
- $h = 60 \tan 35° \approx 42.01$ m
  Solve for h.

EXERCISES (EASY → DIFFICULT)

1. A ladder leans at 60° to the ground and reaches 3 m up a wall. Find the ladder's length.
2. From 50 m away, the angle of elevation to a tower top is 30°. Find the tower's height.
3. A 10 m flagpole casts a shadow when the sun's elevation is 45°. Find the shadow length.
4. In △ABC, $a = 8$, $A = 60°$, $B = 40°$. Use the sine rule to find b.
5. Find the area of △ABC with $b = 6$, $c = 8$, $A = 30°$.
6. From a boat, the top of a 40 m lighthouse has an elevation of 25°. How far is the boat from its base?
7. In △ABC, $b = 7$, $c = 9$ and $A = 50°$. Use the cosine rule to find a.
8. A surveyor 60 m from a tower measures its top at 35° elevation. Find the tower's height.
9. In △ABC, $a = 5$, $b = 7$, $c = 8$. Find angle A using the cosine rule.
10. A plane flies 200 km on a bearing of 060°, then 150 km on a bearing of 120°. How far is it from the start?

End of Chapter 12 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

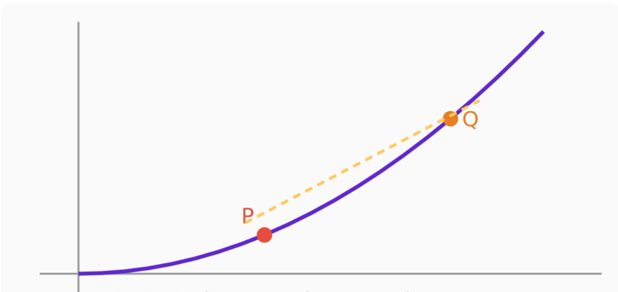
1. Write the expansion of $\sin(A + B)$.
2. Write the three forms of $\cos 2\theta$.
3. Solve $\tan\theta = 1$ for $0° \le \theta \le 360°$.
4. In △ABC, $a = 8$, $A = 60°$, $B = 40°$. Use the sine rule to find b.
5. Simplify $\cos 50°\cos 10° - \sin 50°\sin 10°$.
6. A wheel's spoke makes angle θ with $\sin\theta = 8/17$ (θ acute). Find $\sin 2\theta$ for the doubled angle.
7. Solve $\sin 2\theta = \sin\theta$ for $0° \le \theta \le 360°$.
8. A surveyor 60 m from a tower measures its top at 35° elevation. Find the tower's height.
9. Simplify $\cos(90° - \theta)$.
10. If $\cos\theta = 12/13$ (θ acute), find $\cos 2\theta$.

CHAPTER 11

Calculus

12.C1 Differentiation from first principles
12.C2 Power rule
12.C3 Differentiating polynomials
12.C4 Tangents and gradients
12.C5 Second derivative and concavity
12.C6 Cubic polynomials
12.C7 Sketching cubics
12.C8 Optimisation (max/min word problems)
12.C9 Cubic Quest — interactive review

12.C1 Differentiation from first principles



As Q→P, the secant becomes the tangent.

DEFINITIONS

Derivative	The instantaneous rate of change of a function — the gradient of the tangent at a point.
First principles	Finding the derivative using the limit $f'(x) = \lim(h\to0)\ [f(x+h) - f(x)] / h$.
Limit (h→0)	The value an expression approaches as h gets infinitely close to zero.

KEY FORMULAS

- **First principles definition:** $f'(x) = \lim_{h\to0} [\, f(x+h) - f(x) \,] / h$
- **Geometric meaning:** the derivative is the gradient of the tangent at a point
- **Strategy:** expand $f(x+h)$, subtract $f(x)$, simplify, divide by h, then let $h \to 0$
- **Cancel the h:** after dividing, every term should have an h that cancels (or terms that vanish as $h \to 0$)
- **Newton & Leibniz** independently discovered this in the 1660s–1670s
- **The point where Q → P:** secant becomes tangent

WORKED EXAMPLES

Example 1: Find f'(x) for f(x) = x² from first principles.

▸ $f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$.
Square the bracket carefully.
▸ $f(x+h) - f(x) = (x^2 + 2xh + h^2) - x^2 = \mathbf{2xh + h^2}$.
x^2 cancels — what's left contains h.
▸ Divide by h: $(2xh + h^2) / h = \mathbf{2x + h}$.
Every term had an h to cancel.
▸ Let $h \to 0$: $\mathbf{f'(x) = 2x}$.
The h disappears, leaving the derivative.

Example 2: Find f'(x) for f(x) = 3x by first principles.

▸ $f(x+h) = 3(x+h) = 3x + 3h$.
Distribute.
▸ $f(x+h) - f(x) = (3x + 3h) - 3x = 3h$.
Subtract.
▸ Divide by h: $3h/h = 3$.
Simplify.
▸ Let $h \to 0$: $\mathbf{f'(x) = 3}$.
Linear function: gradient is constant.

Applied example — Velocity from first principles

A stone’s position is $s(t) = t^2$ m. Find its velocity $s'(t)$ from first principles.

▸ $s'(t) = \lim(h\to0)[(t+h)^2 - t^2]/h$
Definition of the derivative.
▸ $= \lim(h\to0)(2t + h)$
Expand and simplify.
▸ $= 2t$ m/s
Let $h \to 0$.

EXERCISES (EASY → DIFFICULT)

1. State the definition of the derivative from first principles.
2. Find $f'(x)$ for $f(x) = 3$ from first principles.
3. Find $f'(x)$ for $f(x) = 3x$ from first principles.
4. Find $f'(x)$ for $f(x) = x^2$ from first principles.
5. Find $f'(x)$ for $f(x) = x^2 + 1$ from first principles.
6. Find $f'(x)$ for $f(x) = 2x^2$ from first principles.
7. Find $f'(x)$ for $f(x) = x^2 - 3x$ from first principles.
8. Find $f'(x)$ for $f(x) = x^3$ from first principles.
9. Find $f'(x)$ for $f(x) = -x^2$ from first principles.
10. Find $f'(x)$ for $f(x) = 1/x$ from first principles.

12.C2 Power rule (differentiating x^n)

$$\frac{d}{dx}(x^{3}) = 3 \cdot x^{2}$$

Bring exponent down, subtract 1 from the power.

DEFINITIONS

Power rule	A shortcut for differentiating x^n: bring the power down and reduce it by one.
Exponent	The power n in x^n — it becomes the multiplier when differentiating.

KEY FORMULAS

- **Power rule:** $d/dx\,(x^{n}) = n \cdot x^{n-1}$
- **Constant rule:** $d/dx\,(k) = 0$ (any constant has zero gradient)
- **Coefficient rule:** $d/dx\,(k \cdot x^{n}) = k \cdot n \cdot x^{n-1}$
- **Recipe:** bring the exponent down as a coefficient, then subtract 1 from the exponent
- **x^1:** $d/dx\,(x) = 1$; **$x^0 = 1$:** $d/dx\,(1) = 0$
- **Negative powers:** $d/dx\,(x^{-1}) = -x^{-2} = -1/x^2$

WORKED EXAMPLES

Example 1: Differentiate $f(x) = x^4$.

▸ **Bring 4 down:** coefficient becomes 4.
Apply power rule.
▸ **Subtract 1:** new exponent is 3.
Power decreases by 1.
▸ $f'(x) = \mathbf{4x^3}$.
Final form.

Example 2: Differentiate $f(x) = 5x^2$.

▸ **Coefficient stays**; apply power rule to x^2.
Pull out the 5.
▸ $f'(x) = 5 \cdot (2x) = \mathbf{10x}$.
Bring 2 down, subtract 1, multiply.

Example 3: Differentiate $f(x) = 7$.

▸ **7 is a constant.**
Constants have zero gradient.
▸ $f'(x) = \mathbf{0}$.
Always zero for any constant.

Applied example — Velocity of a car

A car's distance is $s(t) = 4t^2$ m. Find its velocity function.

- ▸ $s'(t) = 8t$
  Power rule on each term.
- ▸ Velocity = $8t$ m/s
  The derivative of distance is velocity.

EXERCISES (EASY → DIFFICULT)

1. Differentiate $f(x) = x^4$.
2. Differentiate $f(x) = x^7$.
3. Differentiate $f(x) = 5x^2$.
4. Differentiate $f(x) = 3x^5$.
5. Differentiate $f(x) = 7$.
6. The distance of a car is $s(t) = 4t^2$ metres. Its velocity is $s'(t)$. Find the velocity function.
7. Differentiate $f(x) = x^{(-2)}$.
8. Differentiate $f(x) = \sqrt{x}$ ($= x^{½}$).
9. A tank drains so the volume is $V(t) = 4/t^2$ ($= 4t^{(-2)}$). Find the rate of change $V'(t)$.
10. Differentiate $f(x) = 2x^{(3/2)}$.

12.C3 Differentiating polynomials

$$\frac{d}{dx}\left(x^{3}\right) = 3 \cdot x^{2}$$

Bring exponent down, subtract 1 from the power.

DEFINITIONS

Term-by-term	Differentiate each term of the polynomial separately, then add the results.
Constant rule	The derivative of a constant is 0 — it has no rate of change.

KEY FORMULAS

- **Sum rule:** $d/dx\,[\,f(x) + g(x)\,] = f'(x) + g'(x)$
- **Difference rule:** $d/dx\,[\,f(x) - g(x)\,] = f'(x) - g'(x)$
- **Coefficient rule:** $d/dx\,[\,k\cdot f(x)\,] = k\cdot f'(x)$
- **Constant rule:** d/dx (constant) = 0
- **Strategy:** differentiate term by term, applying the power rule to each
- **Notation:** $f'(x)$ or dy/dx or $D_x(y)$ all mean the derivative

WORKED EXAMPLES

Example 1: Differentiate $f(x) = 3x^2 + 2x - 5$.

- ▸ **$3x^2$** → $3\cdot(2x) = 6x$.
  Power rule on first term.
- ▸ **$2x$** → $2\cdot(1) = 2$.
  x^1 differentiates to 1.
- ▸ **-5** → 0.
  Constant has zero gradient.
- ▸ $f'(x) = \mathbf{6x + 2}$.
  Combine.

Example 2: Differentiate $f(x) = x^3 - 4x^2 + 7x - 1$.

- ▸ $x^3 \to 3x^2$, $-4x^2 \to -8x$, $7x \to 7$, $-1 \to 0$.
  Each term separately.
- ▸ $f'(x) = \mathbf{3x^2 - 8x + 7}$.
  Combine.

Applied example — Velocity of a thrown ball

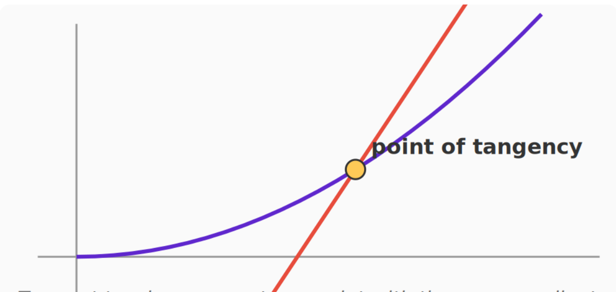
A ball's height is $h(t) = 20t - 5t^2$ m. Find the velocity at $t = 1$ s.

- ▸ $h'(t) = 20 - 10t$
  Differentiate term by term.
- ▸ $h'(1) = 20 - 10 = 10$ m/s
  Substitute $t = 1$.

EXERCISES (EASY → DIFFICULT)

1. Differentiate $f(x) = 3x^2 + 2x - 5$.
2. Differentiate $f(x) = x^3 - 4x^2 + 7x - 1$.
3. Differentiate $f(x) = 5x^2 - 2x + 1$.
4. Differentiate $f(x) = x^4 - x^2$.
5. Differentiate $f(x) = (x + 2)(x - 3)$.
6. A ball's height is $h(t) = 20t - 5t^2$ metres. Find its velocity function $h'(t)$.
7. Using $h(t) = 20t - 5t^2$, find the velocity at $t = 1$ s.
8. A company's profit is $P(x) = x^3 - 3x$ (in R1000s). Find the marginal profit $P'(x)$.
9. Differentiate $f(x) = (2x - 1)^2$.
10. Find $f'(2)$ for $f(x) = x^3 - 3x$.

12.C4 Tangents and gradients



Tangent touches curve at one point with the same gradient.

DEFINITIONS

Tangent	A straight line that touches the curve at one point with the same gradient as the curve there.
Gradient at a point	The value of f'(x) at that x — the slope of the tangent.

KEY FORMULAS

- **Gradient at a point:** the value of f'(x) at that x-value
- **Equation of tangent at (a, f(a)):** $y - f(a) = f'(a) \cdot (x - a)$
- **Strategy:** (1) find f'(x), (2) evaluate f'(a) for the gradient, (3) find the y-coordinate f(a), (4) write the line equation
- **Tangent** touches the curve at one point and has the same gradient
- **Normal:** perpendicular to the tangent, gradient $= -1 / f'(a)$
- **Stationary point** $\iff f'(x) = 0$ (gradient is zero)

WORKED EXAMPLES

Example 1: Find the gradient of $f(x) = x^2$ at $x = 3$.

▸ **Step 1:** $f'(x) = 2x$.
Power rule.
▸ **Step 2:** $f'(3) = 2(3) = \mathbf{6}$.
Substitute $x = 3$.

Example 2: Find the equation of the tangent to $f(x) = x^2$ at $x = 1$.

▸ **Step 1:** $f'(x) = 2x$, so $f'(1) = 2$.
Gradient at $x = 1$.
▸ **Step 2:** $f(1) = 1$, so the point is (1, 1).
Y-coordinate of the point of tangency.
▸ **Step 3:** $y - 1 = 2(x - 1) \implies y = 2x - 1$.
Use point-slope, then simplify.
▸ **Tangent: $y = 2x - 1$.**
Final equation.

Applied example — Slope of a path

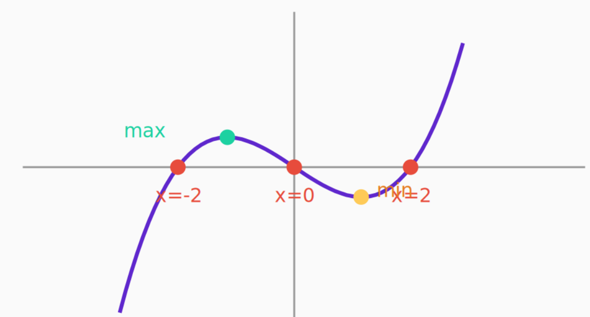
A ball follows $y = x^2$. Find the slope of its path at $x = 2$.

- ▸ $y' = 2x$
  Power rule.
- ▸ At $x = 2$, slope $= 4$
  Substitute.

EXERCISES (EASY → DIFFICULT)

1. Find the gradient of $f(x) = x^2$ at $x = 3$.
2. Find the gradient of $f(x) = x^3$ at $x = 2$.
3. Where is the gradient of $f(x) = x^2 - 6x$ equal to 0?
4. Find the equation of the tangent to $f(x) = x^2$ at $x = 1$.
5. Find the equation of the tangent to $f(x) = x^2$ at $x = 3$.
6. A ball's path is $y = x^2$ where x is horizontal distance. Find the slope of the path at $x = 2$.
7. Find the tangent to $f(x) = x^3 - 3x$ at $x = 2$.
8. At what point does $f(x) = x^2$ have a tangent of gradient 8?
9. Find the equation of the normal to $f(x) = x^2$ at $x = 1$.
10. Find the points on $f(x) = x^3 - 12x$ where the tangent is horizontal.

12.C5 Second derivative and concavity



Roots (red) and stationary points (green/yellow)

DEFINITIONS

Second derivative f″(x)	The derivative of the derivative — it measures how the gradient itself is changing.
Concave up	Where $f''(x) > 0$ — the curve bends upward like a cup.
Point of inflection	Where concavity changes and $f''(x) = 0$.

KEY FORMULAS

- **Second derivative:** $f''(x)$ = derivative of $f'(x)$
- **Notation:** $f''(x)$ or d^2y/dx^2 or $D^2_x(y)$
- **Concave up** (∪-shape) when $f''(x) > 0$
- **Concave down** (∩-shape) when $f''(x) < 0$
- **Inflection point** where $f''(x) = 0$ and the concavity changes
- **Test for max/min** at a stationary point: if $f''(a) > 0$ → MIN; if $f''(a) < 0$ → MAX

WORKED EXAMPLES

Example 1: Find $f''(x)$ for $f(x) = x^3$.

- **Step 1:** $f'(x) = 3x^2$.

Power rule once.

- **Step 2:** $f''(x) = 6x$.

Power rule again.

Example 2: Is $f(x) = x^2$ concave up or down at $x = 0$?

- **Step 1:** $f'(x) = 2x$, $f''(x) = 2$.

Differentiate twice.

- **Step 2:** $f''(0) = 2 > 0$.

Positive everywhere, including at 0.

- Concave **UP** ($\cup$).

Parabola opens upwards.

Applied example — Acceleration of a car

A car's position is $s(t) = t^3 - 6t^2$. Find when the acceleration is zero.

- $s'(t) = 3t^2 - 12t$, $s''(t) = 6t - 12$

Differentiate twice.

- $6t - 12 = 0 \rightarrow t = 2$ s

Acceleration is the second derivative.

EXERCISES (EASY → DIFFICULT)

1. Find $f''(x)$ for $f(x) = x^3$.
2. Find $f''(x)$ for $f(x) = x^4$.
3. Find $f''(x)$ for $f(x) = x^3 - 6x^2$.
4. Find $f''(x)$ for $f(x) = 2x^3 - x$.
5. For $f(x) = x^3 - 6x^2$, find where $f''(x) = 0$ (the point of inflection's x-value).
6. A car's position is $s(t) = t^3 - 6t^2$. Its acceleration is $s''(t)$. Find the acceleration function.
7. Using $s(t) = t^3 - 6t^2$, find when the acceleration is zero.
8. Where is $f(x) = x^3 - 3x^2$ concave up?
9. A stationary point of $f(x) = x^2 - 4x$ is at $x = 2$. Use the second derivative to classify it.
10. Classify the stationary point of $f(x) = -x^2 + 6x$.

12.C6 Cubic polynomials

$f(x) = x^3 - 6x^2 + 11x - 6$

Try $x = 1$: $1 - 6 + 11 - 6 =$ **0** ✓

$\Longrightarrow$ $(x - 1)$ is a factor.

DEFINITIONS

Cubic	A degree-3 polynomial of the form $ax^3 + bx^2 + cx + d$.
Factor theorem	$(x - a)$ is a factor if $f(a) = 0$ — used to factor cubics.

KEY FORMULAS

- **Cubic:** $f(x) = ax^3 + bx^2 + cx + d$ (degree 3)
- **Factor theorem:** if $f(k) = 0$, then $(x - k)$ is a factor of $f(x)$
- **Remainder theorem:** $f(k)$ = remainder when $f(x)$ is divided by $(x - k)$
- **Strategy:** test small integer values of k (e.g. $\pm1, \pm2, \pm3$) to find roots
- **Then divide** the cubic by $(x - k)$ using long division to get a quadratic
- **Solve the quadratic** using factoring or the quadratic formula to find remaining roots

WORKED EXAMPLES

Example 1: Solve $f(x) = x^3 - 6x^2 + 11x - 6 = 0$.

▸ **Test $x = 1$:** $1 - 6 + 11 - 6 = 0$.
First root found.

▸ **$(x - 1)$ is a factor.** Long-divide: $x^3 - 6x^2 + 11x - 6 \div (x - 1) = x^2 - 5x + 6$.
Reduces cubic to quadratic.

▸ **Factor the quadratic:** $x^2 - 5x + 6 = (x - 2)(x - 3)$.
Two more roots.

▸ **Roots: $x = 1, 2, 3$.**
All three.

Example 2: Find one root of $f(x) = x^3 - x^2 - 4x + 4$.

▸ **Test $x = 1$:** $1 - 1 - 4 + 4 = 0$.
$x = 1$ is a root.

▸ **$(x - 1)$ is a factor.**
Confirmed by factor theorem.

Applied example — Roots of a ride profile

A ride dip is $f(x) = x^3 - 3x^2 + 4$. Show $x = 2$ is a root and factorise.

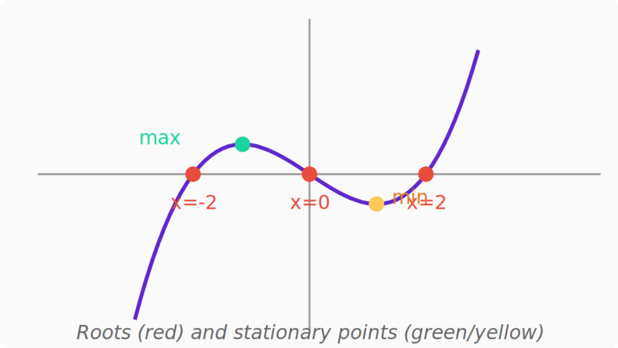
▸ $f(2) = 8 - 12 + 4 = 0$
So $(x - 2)$ is a factor.

▸ $f(x) = (x - 2)^2(x + 1)$
Divide and factorise.

EXERCISES (EASY → DIFFICULT)

1. Show that $x = 1$ is a root of $f(x) = x^3 - 6x^2 + 11x - 6$.
2. Factorise $f(x) = x^3 - 6x^2 + 11x - 6$.
3. Find the x-intercepts of $f(x) = x^3 - 4x$.
4. Find the y-intercept of $f(x) = x^3 - 2x^2 + x - 5$.
5. Find the x-values of the stationary points of $f(x) = x^3 - 3x$.
6. A roller-coaster dip is modelled by $f(x) = x^3 - 3x^2 + 4$. Show that $x = 2$ is a root and factorise.
7. For $f(x) = x^3 - 6x^2 + 9x$, find the turning points.
8. How many distinct real roots has $f(x) = x^3 - 3x + 2$?
9. A cubic has roots $-1, 2, 3$ and leading coefficient 1. Write $f(x)$.
10. Determine k so that $x = 2$ is a root of $f(x) = x^3 - kx + 2$.

12.C7 Sketching cubics



Roots (red) and stationary points (green/yellow)

DEFINITIONS

Intercepts	Where the curve cuts the axes: y-intercept (x=0) and x-intercepts (f(x)=0).
Stationary point	A turning point where f'(x) = 0 (local max or min).
Turning point	A point where the curve changes from rising to falling, or vice versa.

KEY FORMULAS

- **Roots (x-intercepts):** solve $f(x) = 0$
- **Y-intercept:** $f(0)$ = the constant term d
- **Stationary points:** solve $f'(x) = 0$; substitute back to get y-coordinate
- **Classify each:** use $f''(x)$ — positive at min, negative at max
- **Inflection point:** solve $f''(x) = 0$
- **End behaviour:** if $a > 0$, curve goes from $-\infty$ (left) to $+\infty$ (right). If $a < 0$, opposite.

WORKED EXAMPLES

Example 1: Find the y-intercept of $f(x) = x^3 - 6x^2 + 11x - 6$.

▸ **Set $x = 0$:** $f(0) = 0 - 0 + 0 - 6 = -6$.

Just read off the constant term.

▸ **Y-intercept: $(0, -6)$.**

Cross the y-axis at $y = -6$.

Example 2: Find stationary points of $f(x) = x^3 - 3x$.

- **Step 1:** $f'(x) = 3x^2 - 3$.
  Differentiate.
- **Step 2:** Set $f'(x) = 0$: $3x^2 = 3 \Longrightarrow x^2 = 1 \Longrightarrow x = \pm 1$.
  Solve for the critical points.
- **Step 3:** y-values: $f(1) = 1 - 3 = -2$; $f(-1) = -1 + 3 = 2$.
  Plug back into the original function.
- **Stationary points: $(1, -2)$ and $(-1, 2)$**.
  Two stationary points.

Applied example — Turning points of a stall’s profit

A stall’s profit is $P(x) = x^3 - 6x^2 + 9x$ (R1000s). Find its turning points.

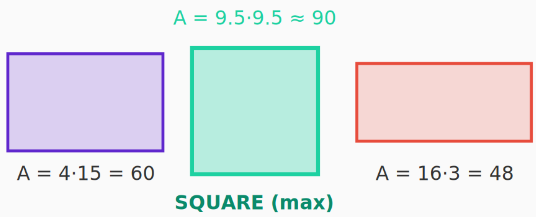
- $P'(x) = 3x^2 - 12x + 9 = 0$
  Set the derivative to zero.
- $x = 1$ or $x = 3$
  Solve the quadratic.
- Turning points $(1, 4)$ and $(3, 0)$
  Substitute back into P.

EXERCISES (EASY → DIFFICULT)

1. Find the y-intercept of $f(x) = x^3 - 3x^2 + 2$.
2. Find the stationary points of $f(x) = x^3 - 3x$.
3. Classify the stationary points of $f(x) = x^3 - 3x$.
4. Find the point of inflection of $f(x) = x^3 - 3x$.
5. Find the x-intercepts of $f(x) = x^3 - 6x^2 + 9x$.
6. Sketch help: is $f(x) = x^3$ increasing or decreasing throughout?
7. For $f(x) = -x^3 + 3x$, find and classify the turning points.
8. The profit (R1000s) of a stall is $P(x) = x^3 - 6x^2 + 9x$ for $0 \le x \le 4$ units sold. Find the turning points of P.
9. Find a cubic with a local max at $(-1, 2)$ and a local min at $(1, -2)$, passing through the origin.
10. For which x is $f(x) = x^3 - 3x$ increasing?

12.C8 Optimisation

Optimisation (max/min word problems)



Same perimeter (38) — square gives the max area.

DEFINITIONS

Optimisation	Finding the maximum or minimum value of a quantity using calculus.
Stationary point	Where $f'(x) = 0$ — a candidate for the maximum or minimum.

KEY FORMULAS

- **Strategy:** express the quantity to optimise as a function of ONE variable
- **Then differentiate** and set the derivative to 0
- **Solve** for the critical x-value(s)
- **Confirm** max or min using $f''(x)$ (positive $\Longrightarrow$ min, negative $\Longrightarrow$ max)
- **Common context:** max area, min surface area, max volume, max profit, min cost
- **Final step:** evaluate the original quantity at the optimal x to get the actual answer

WORKED EXAMPLES

Example 1: A rectangle has perimeter 20. What dimensions maximise its area?

▸ **Setup:** let length = x, width = w. Then $2x + 2w = 20 \Longrightarrow w = 10 - x$.

Express width in terms of length using the perimeter.

▸ **Area:** $A = x \cdot w = x(10 - x) = 10x - x^2$.

Now A is a function of one variable.

▸ **Differentiate:** $A'(x) = 10 - 2x$. Set $A'(x) = 0$: $x = 5$.

Find stationary point.

▸ **Verify:** $A''(x) = -2 < 0 \rightarrow$ MAX.

Confirms it's a maximum.

▸ **Answer:** length = 5, width = 5 → it's a SQUARE. Maximum area = **25**.

Square always maximises area for a fixed perimeter.

Example 2: Find the maximum value of $A = 12x - x^2$ (for $x > 0$).

▸ **Differentiate:** $A'(x) = 12 - 2x$.

Power rule.

▸ **Set to 0:** $12 - 2x = 0 \Longrightarrow x = 6$.

Stationary point.

▸ **Verify max:** $A''(x) = -2 < 0 \rightarrow$ MAX.

Negative second derivative confirms maximum.

▸ **Maximum value:** $A(6) = 72 - 36 =$ **36**.

Substitute $x = 6$ back.

Applied example — Largest kraal area

A farmer has 100 m of fence for a rectangular kraal against a wall (3 sides). Find the dimensions for maximum area.

- ▸ $A = x(100 - 2x)$
  Width x, the wall replaces one length.
- ▸ $A'(x) = 100 - 4x = 0 \rightarrow x = 25$
  Maximise.
- ▸ 25 m × 50 m, area 1 250 m^2
  The largest area.

EXERCISES (EASY → DIFFICULT)

1. $A(x) = 6x - x^2$. Find the value of x that maximises A.
2. Find the maximum value of $A(x) = 6x - x^2$.
3. Find the minimum of $f(x) = x^2 - 6x + 10$.
4. The sum of two numbers is 20. Maximise their product.
5. A rectangle has a perimeter of 20 m. Find the dimensions that maximise its area.
6. Find that rectangle's maximum area.
7. A farmer has 100 m of fencing for a rectangular kraal against a wall (only 3 sides need fencing). Find the dimensions for maximum area.
8. Find that kraal's maximum area.
9. An open box is made from card; its volume is $V(x) = x(10 - 2x)^2$ $(0 < x < 5)$. Find x that maximises the volume.
10. A cool-drink can must hold a fixed volume. Briefly, what is the calculus method to minimise the metal used?

12.C9 Cubic Quest (interactive game)

Cubic Quest — interactive review

DEFINITIONS

Cubic Quest	An interactive game that walks you through factor theorem, long division, cubic factoring, sketching, and first principles — all in one challenge.

KEY FORMULAS

- **Covers:** factor theorem · long division · solving cubics · function analysis · first principles
- **How to play:** Tap the launch button below and work through each screen.

WORKED EXAMPLES

Example 1: Launch the Cubic Quest game below.

- ▸ Work through each screen at your own pace.
  Tap items in order; the game tracks progress.

Applied example — Inflection of a ride

A ride’s height is $f(x) = x^3 - 6x^2 + 5$. Find its point of inflection.

- $f''(x) = 6x - 12 = 0 \to x = 2$
  Inflection where the second derivative is zero.
- $f(2) = 8 - 24 + 5 = -11 \to (2, -11)$
  Substitute back.

EXERCISES (EASY → DIFFICULT)

1. What is the derivative of $f(x) = x^3$?
2. Find the stationary points of $f(x) = x^3 - 3x$.
3. Where does $f(x) = x^3 - 3x$ have a local maximum?
4. Factorise $x^3 - x$.
5. Solve $x^3 - 4x = 0$.
6. Find $f''(x)$ for $f(x) = x^3 - 6x^2$.
7. A ride’s height is $f(x) = x^3 - 6x^2 + 5$. Find its point of inflection.
8. For what x is $f(x) = x^3 - 3x$ increasing?
9. Find the tangent to $f(x) = x^3$ at $x = 1$.
10. A cubic has both a local max and a local min. How many stationary points does it have?

End of Chapter 11 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. State the definition of the derivative from first principles.
2. Differentiate $f(x) = x^7$.
3. Differentiate $f(x) = 5x^2 - 2x + 1$.
4. Find the equation of the tangent to $f(x) = x^2$ at $x = 1$.
5. For $f(x) = x^3 - 6x^2$, find where $f''(x) = 0$ (the point of inflection’s x-value).
6. A roller-coaster dip is modelled by $f(x) = x^3 - 3x^2 + 4$. Show that $x = 2$ is a root and factorise.
7. For $f(x) = -x^3 + 3x$, find and classify the turning points.
8. Find that kraal’s maximum area.
9. Find the tangent to $f(x) = x^3$ at $x = 1$.
10. Find $f'(x)$ for $f(x) = 1/x$ from first principles.

CHAPTER 12

Past Papers — Algebra

Past Past-paper equations — quadratic, surd & simultaneous
Past Past-paper inequalities — quadratic inequalities
Past Past-paper proofs — algebraic manipulation

Past Papers — Algebra: Equations

Past-paper equations — quadratic, surd & simultaneous

DEFINITIONS

Zero-product rule	If a product of factors equals zero, then at least one factor must be zero.
Standard form	A quadratic written as a x squared plus b x plus c equals zero, so the coefficients are easy to read off.
Extraneous solution	A value introduced by squaring both sides that does not satisfy the original equation, so it must be rejected.

KEY FORMULAS

- **Quadratic formula:** $x = (-b \pm \sqrt{b^2 - 4ac}) \div 2a$
- **Zero-product rule:** if $A \cdot B = 0$ then $A = 0$ or $B = 0$
- **Surd equations:** isolate the root, square both sides, then *check* every answer in the original

WORKED EXAMPLES

Example 1: NSC P1 · Q1.1.1 Solve for x: $2x(3x + 4) = 0$ *(2)*

▸ $2x(3x + 4) = 0$

A product equals zero, so use the zero-product rule: one factor must be zero.

▸ $2x = 0$ or $3x + 4 = 0$

Set each factor equal to zero separately.

▸ $\mathbf{x = 0}$ or $\mathbf{x = -4/3}$

Solve each small equation. These are the two roots.

Example 2: NSC P1 · Q1.1.2 Solve for x: $2x^2 - 4x = -1$ (correct to TWO decimal places) *(4)*

▸ $2x^2 - 4x + 1 = 0$

Move every term to one side to get standard form.

▸ $a = 2, \quad b = -4, \quad c = 1$

Read off the coefficients for the quadratic formula.

▸ $x = (-(-4) \pm \sqrt{((-4)^2 - 4(2)(1))}) \div (2 \cdot 2)$

Substitute into the quadratic formula.

▸ $x = (4 \pm \sqrt{8}) \div 4$

Simplify the discriminant: sixteen minus eight is eight.

▸ $\mathbf{x \approx 1{,}71}$ or $\mathbf{x \approx 0{,}29}$

Evaluate both and round to two decimal places, as instructed.

Example 3: NSC P1 · Q1.2.1 Given $\sqrt{(x - 2)} = 4 - x$. Without solving, show the solution lies in $2 \le x \le 4$. *(2)*

▸ Radicand: $x - 2 \ge 0 \;\rightarrow\; x \ge 2$

A square root is only defined when what is under the root is not negative.

▸ Right side: $4 - x \ge 0 \;\rightarrow\; x \le 4$

A square root is never negative, so the right-hand side cannot be negative either.

▸ ▯ $\mathbf{2 \le x \le 4}$

Combine the two restrictions — any solution must lie in this interval.

Example 4: NSC P1 · Q1.2.2 Solve the equation $\sqrt{(x - 2)} = 4 - x$. *(4)*

▸ $x - 2 = (4 - x)^2 = 16 - 8x + x^2$

Square both sides to remove the square root.

▸ $x^2 - 9x + 18 = 0$

Collect all terms on one side and tidy up.

▸ $(x - 6)(x - 3) = 0 \;\rightarrow\; x = 6$ or $x = 3$

Factorise and solve.

▸ $\mathbf{x = 3}$ (reject $x = 6$)

From Q1.2.1 the solution must satisfy $2 \le x \le 4$, so $x = 6$ is extraneous and is rejected.

Example 5: NSC P1 · Q1.3 Solve simultaneously: $3x + y = 2$ and $y^2 = 2x^2 - 1$ *(6)*

▸ From the linear equation: $y = 2 - 3x$

Make y the subject of the simpler equation.

▸ $(2 - 3x)^2 = 2x^2 - 1$

Substitute into the second equation.

▸ $4 - 12x + 9x^2 = 2x^2 - 1 \;\rightarrow\; 7x^2 - 12x + 5 = 0$

Expand and bring to standard form.

▸ $(7x - 5)(x - 1) = 0 \;\rightarrow\; x = 5/7$ or $x = 1$

Factorise and solve for x.

▸ $\mathbf{x = 5/7, y = -1/7}$ or $\mathbf{x = 1, y = -1}$

Substitute each x back into $y = 2 - 3x$ to find the matching y.

Applied example — When does the ball land?

A ball's height is $h = 20t - 5t^2$ m. Find the times when it is on the ground.

▸ $20t - 5t^2 = 0$

Set the height to zero.

▸ $5t(4 - t) = 0 \rightarrow t = 0$ s or $t = 4$ s

Factorise and solve.

EXERCISES (EASY → DIFFICULT)

1. Solve $x^2 - 5x + 6 = 0$.
2. Solve $x^2 - 9 = 0$.
3. Solve $2x(3x + 4) = 0$.
4. Solve $x^2 + 2x - 8 = 0$.
5. Solve $2x^2 - 4x + 1 = 0$ (to 2 dp).
6. A ball's height is $h = 20t - 5t^2$ metres. Find the times when it is on the ground ($h = 0$).
7. A rectangular garden is 3 m longer than it is wide and has area 40 m^2. Find its width.
8. Solve $\sqrt{(x - 2)} = 4 - x$, and say why one solution is rejected.
9. Solve simultaneously: $3x + y = 2$ and $y^2 = 2x^2 - 1$.
10. Solve $x^4 - 5x^2 + 4 = 0$.

Past Papers — Algebra: Inequalities

Past-paper inequalities — quadratic inequalities

DEFINITIONS

Critical values	The x-values where the expression equals zero; they split the number line into regions to test.
Region test	For an upward parabola the expression is greater than or equal to zero outside its roots, and less than or equal to zero between them.

KEY FORMULAS

- **Quadratic inequality:** factorise → critical values → test regions
- **Upward parabola ($a > 0$):** expression ≥ 0 *outside* the roots; ≤ 0 *between* them

WORKED EXAMPLES

Example 1: NSC P1 · Q1.1.3 Solve for x: $(x - 2)^2 \geq 1$ *(4)*

▸ $(x - 2)^2 \geq 1 \rightarrow x^2 - 4x + 4 \geq 1$

Expand the square on the left.

▸ $x^2 - 4x + 3 \geq 0$

Bring the 1 across so one side is zero.

▸ $(x - 3)(x - 1) \geq 0$

Factorise the quadratic.

▸ Critical values: $x = 1$ and $x = 3$

These are where the expression equals zero; they split the number line.

▸ $\mathbf{x \leq 1}$ **or** $\mathbf{x \geq 3}$

The parabola opens upward, so it is greater than or equal to zero outside the roots — see the number line.

Applied example — When is the firm profitable?

A profit model is $P = (x - 2)(x - 6)$ (R1000s). For which production levels is $P > 0$?

- ▸ $(x - 2)(x - 6) > 0$

  Profit positive.
- ▸ $x < 2$ or $x > 6$

  Product positive outside the roots (upward parabola).

EXERCISES (EASY → DIFFICULT)

1. Solve $x - 3 > 0$.
2. Solve $2x + 1 \le 5$.
3. Solve $(x - 1)(x - 3) > 0$.
4. Solve $(x + 2)(x - 4) < 0$.
5. Solve $x^2 - 9 \le 0$.
6. A profit (R1000s) model gives $P = (x - 2)(x - 6)$. For which production levels x is the firm profitable ($P > 0$)?
7. A projectile stays above 0 height while $-x^2 + 4x > 0$. For which x is this true?
8. Solve $(x - 2)^2 \ge 1$.
9. Solve $x^2 \ge 4x$.
10. Solve $(x + 1)/(x - 2) \ge 0$ ($x \ne 2$).

Past Papers — Algebra: Proofs & Identities

Past-paper proofs — algebraic manipulation

DEFINITIONS

Difference of squares	The identity: a plus b, times a minus b, equals a squared minus b squared.
Elimination	Adding or subtracting two equations to remove one variable.

KEY FORMULAS

- **Difference of squares:** $(a + b)(a - b) = a^2 - b^2$
- **Elimination:** add the equations to remove a subtracted term; subtract to remove an added term

WORKED EXAMPLES

Example 1: NSC P1 · Q1.4 If $r + 2s = a$ and $r - 2s = b$, prove that $rs = (a^2 - b^2) / 8$. *(4)*

- ▸ Add the equations: $2r = a + b \rightarrow r = (a + b)/2$

  Adding eliminates the s terms.
- ▸ Subtract the equations: $4s = a - b \rightarrow s = (a - b)/4$

  Subtracting eliminates the r terms.
- ▸ $rs = (a + b)/2 \cdot (a - b)/4 = (a + b)(a - b)/8$

  Multiply r and s together.
- ▸ $\mathbf{rs = (a^2 - b^2)/8}$

  Apply the difference-of-squares identity. This is exactly what we had to prove.

Applied example — A widened room

A square room of side x is widened by 2 m on one pair of walls and shortened by 2 m on the other. Show the new area is 4 m^2 less.

- ▸ New area = $(x + 2)(x - 2)$

  Multiply the new dimensions.
- ▸ $= x^2 - 4$

  Difference of squares — exactly 4 m^2 less than x^2.

EXERCISES (EASY → DIFFICULT)

1. Simplify $(a + b)(a - b)$.
2. Expand $(a + b)^2$.
3. Expand $(a - b)^2$.
4. If $r + 2s = a$ and $r - 2s = b$, find r in terms of a and b.
5. Using the same equations, find s.
6. Prove that the sum of any two consecutive integers is odd.
7. A square room of side x is widened by 2 m and shortened by 2 m. Show the new area is 4 m^2 less than the square's.
8. Prove $(a + b)^2 - (a - b)^2 = 4ab$.
9. Show that $x^2 - 6x + 10 > 0$ for all real x.
10. Prove that the product of two consecutive even numbers is divisible by 8.

End of Chapter 12 — Mixed Exercise

Ten questions across the whole chapter, from easy to difficult.

1. Solve $x^2 - 5x + 6 = 0$.
2. Solve $2x + 1 \le 5$.
3. Expand $(a - b)^2$.
4. Solve $x^2 + 2x - 8 = 0$.
5. Solve $x^2 - 9 \le 0$.
6. Prove that the sum of any two consecutive integers is odd.
7. A rectangular garden is 3 m longer than it is wide and has area 40 m^2. Find its width.
8. Solve $(x - 2)^2 \ge 1$.
9. Show that $x^2 - 6x + 10 > 0$ for all real x.
10. Solve $x^4 - 5x^2 + 4 = 0$.

ANSWERS

Worked solutions to all exercises

Each answer gives the final result first, followed by the key working steps. For fully animated, step-by-step solutions, open the digital edition with the QR code on any page.

CHAPTER 3 — Finance

3.1 Period of an investment

1. n = log 2 / log 1.1 ≈ **7.27 years**.
2. n = log 2 / log 1.08 ≈ **9.01 years**.
3. n = log 1.5 / log 1.12 ≈ **3.58 years**.
4. n = log(12000/8000)/log 1.09 ≈ **4.7 years**.
5. n = log 2 / log 1.06 ≈ **11.9 years**.
6. n = log 2.5 / log 1.11 ≈ **8.78 years**.
7. i = 0.0125; n = log 2.5 / log 1.0125 ≈ **73.76 months** (≈ 6.15 years).
8. i = 0.0225; n = log 2.5 / log 1.0225 ≈ **41.18 quarters** (≈ 10.3 years).
9. i = 0.04; n = log 3 / log 1.04 ≈ **28.01 half-years** (≈ 14.01 years).
10. n = log(50000/30000)/log(1+0.075/12) ≈ **81.99 months** (≈ 6.83 years).

3.2 Annuities (definition)

1. A sequence of equal payments made at equal time intervals.
2. A future-value annuity (a sinking fund).
3. A present-value annuity.
4. At the end of each period.
5. At the beginning of each period.
6. The future-value annuity formula.
7. The present-value annuity formula.
8. The present-value annuity formula.
9. A future-value annuity.
10. Because the loan amount equals the present value of all the future repayments.

3.3 Future value annuity formula

1. F = 1000[($(1.1)^5$−1)/0.1] = **R6,105.10**.
2. F = **R7,243.28**.
3. F = **R22,056.95**.
4. i = 0.01, n = 24 → F = **R5,394.69**.
5. i = 0.0075, n = 36 → F = **R12,345.81**.
6. F = **R39,564.30**.
7. i = 0.0125, n = 60 → F = **R22,143.63**.
8. i = 0.02, n = 20 → F = **R24,297.37**.
9. x = F·i/($(1+i)^n$−1) with i=0.00875, n=48 → x = **R1,685.34**.
10. i=0.0075, n=72 → x = **R2,105.11**.

3.4 Present value annuity formula

1. P = **R3,790.79**.
2. P = **R5,368.07**.
3. P = **R6,728.88**.
4. i=0.01, n=24 → P = **R8,497.35**.
5. P = **R15,885.37**.
6. i=0.0075, n=36 → P = **R18,868.08**.
7. i=0.0125, n=60 → P = **R126,103.78**.
8. i=0.02, n=20 → P = **R19,621.72**.
9. i=0.00875, n=48 → P = **R35,151.61**.
10. i=0.0075, n=300 → P = **R953,292.98**.

3.5 Investment/loan analysis

1. i = 12%/12 = **1% per month**.
2. i = 9%/4 = **2.25% per quarter**.
3. i = 8%/2 = **4% per half-year**.
4. $(1.005)^{12}-1$ = **6.1678%** (so the second, 6.1%, is slightly better).
5. $(1.025)^{4}-1$ = **10.3813%**.
6. $(1.0125)^{12}-1$ = **16.0755%**.
7. 12% monthly → effective 12.6825%, which is less than 12.5% → **take 12.5% annually**.
8. i=0.0075, n=36 → x = **R1,589.99**.
9. i=0.01, n=60 → x = **R2,224.44**.
10. i=11%/12, n=240 → x = **R2,580.47**.

Past Paper — Finance (NSC P1 Q6)

1. $F = 12000(1.08)^{4}$ = **R16,325.87**.
2. n = log 1.6 / log 1.1 ≈ **4.93 years**.
3. $(1.0075)^{12}-1$ = **9.3807%**.
4. F = **R60,339.31**.
5. x = **R2,752.72**.
6. $F = 15000(1.01875)^{24}$ = **R23,426.87**.
7. $2 = (1+i)^{9} \rightarrow i = 2^{1/9}-1$ = **8.006%**.
8. $P = 100000(1+0.08/12)^{-120}$ = **R45,052.35**.
9. x=R2,210.80; balance = present value of the remaining 120 payments = **R163,841.69**.
10. $120000 = 1500[(1-(1.0075)^{-n})/0.0075] \rightarrow n \approx$ **122.63** → **101 payments**.

End of Chapter 3 — Mixed Exercise

1. n = log 2 / log 1.1 ≈ **7.27 years**.
2. A future-value annuity (a sinking fund).
3. F = **R22,056.95**.
4. i=0.01, n=24 → P = **R8,497.35**.
5. $(1.025)^{4}-1$ = **10.3813%**.
6. $F = 15000(1.01875)^{24}$ = **R23,426.87**.
7. i = 0.0125; n = log 2.5 / log 1.0125 ≈ **73.76 months** (≈ 6.15 years).
8. The present-value annuity formula.
9. $x = F\cdot i/((1+i)^{n}-1)$ with i=0.00875, n=48 → x = **R1,685.34**.
10. i=0.0075, n=300 → P = **R953,292.98**.

CHAPTER 5 — Polynomials

5.1 Revision: factor $x^2 - 9$

1. $(x - 3)(x + 3)$.
2. $(x - 5)(x + 5)$.
3. $(2x - 3)(2x + 3)$.
4. $(3x - 4)(3x + 4)$.
5. $(5x - 8y)(5x + 8y)$.
6. Area $= x^2 - 9 =$ $\mathbf{(x - 3)(x + 3)}$ m^2.
7. $\mathbf{(2x - 5)(2x + 5)}$.
8. $(x^2 - 4)(x^2 + 4) =$ $\mathbf{(x - 2)(x + 2)(x^2 + 4)}$.
9. $2(25 - x^2) =$ $\mathbf{2(5 - x)(5 + x)}$.
10. $[(x+1) - 3][(x+1) + 3] =$ $\mathbf{(x - 2)(x + 4)}$.

5.2 Cubic polynomials: degree

1. 3.
2. 1.
3. 5.
4. −2.
5. −1.
6. Degree 2 — a parabola.
7. Degree 3; up to 2 turning points.
8. $2 + 3 = 5$.
9. 2 (a cubic has at most 2 turning points).
10. 4.

5.3 Remainder theorem

1. $f(1) = 0 \rightarrow$ remainder $=$ **0**.
2. $f(2) =$ **7**.
3. $f(1) =$ **5**.
4. $f(-1) =$ **−3**.
5. $f(2) =$ **15**.
6. $P(\frac{1}{2}) = \frac{1}{8} - 3 + 4 =$ **9/8 (= 1.125)**.
7. $C(-2) =$ **−21**.
8. $f(2)=8+2k-6=0 \rightarrow$ $\mathbf{k = -1}$.
9. $f(1)=a=4 \rightarrow$ $\mathbf{a = 4}$.
10. $p+q=2$ and $p-q=0 \rightarrow$ $\mathbf{p = q = 1}$.

5.4 Factor theorem

1. $f(2) = 0 \rightarrow$ **yes**.
2. $f(1) = 0 \rightarrow$ **yes**.
3. $f(-1) = 0 \rightarrow$ **yes**.
4. $f(2) = 7 \neq 0 \rightarrow$ **no**.
5. $f(1)=1+k-4+1=0 \rightarrow$ $\mathbf{k = 2}$.
6. $V(-2) = 0 = 0 \rightarrow$ **(x + 2) is a factor**.
7. $x(x^2 + 3x + 2) =$ $\mathbf{x(x + 1)(x + 2)}$.
8. $f(-2)=-16-2a+4=0 \rightarrow$ $\mathbf{a = -6}$.
9. $(x - 1)(x^2 - x - 6) =$ $\mathbf{(x - 1)(x - 3)(x + 2)}$.
10. $b+c=0$ and $b-c=2 \rightarrow$ $\mathbf{b = 1, c = -1}$.

5.5 Solving cubic equations

1. $x(x-1)(x+1)=0 \rightarrow$ **$x = 0$, 1 or -1**.
2. **$x = 2$**.
3. $x(x-2)(x+2)=0 \rightarrow$ **$x = 0$, 2 or -2**.
4. **$x = 1$, 2 or -3**.
5. $x(x-1)(x-2)=0 \rightarrow$ **$x = 0$, 1 or 2**.
6. $x(x-2)(x+2)=0 \rightarrow x>0 \rightarrow$ **$x = 2$ cm** ($x = 0$ or -2 are rejected).
7. $(x-1)(x-2)(x-3)=0 \rightarrow$ **$x = 1$, 2 or 3**.
8. $(x-1)(x-3)(x+2)=0 \rightarrow$ **$x = 1$, 3 or -2**.
9. $(x+2)(x-1)(x+1)=0 \rightarrow$ **$x = -2$, 1 or -1**.
10. $(x+1)(x-3)(x+2)=0 \rightarrow$ **$x = -1$, 3 or -2**.

End of Chapter 5 — Mixed Exercise

1. $(x - 3)(x + 3)$.
2. 1.
3. $f(1) =$ **5**.
4. $f(2) = 7 \neq 0 \rightarrow$ **no**.
5. $x(x-1)(x-2)=0 \rightarrow$ **$x = 0$, 1 or 2**.
6. Area $= x^2 - 9 =$ **$(x - 3)(x + 3)$** m^2.
7. Degree 3; up to 2 turning points.
8. $f(2)=8+2k-6=0 \rightarrow$ **$k = -1$**.
9. $(x - 1)(x^2 - x - 6) =$ **$(x - 1)(x - 3)(x + 2)$**.
10. $(x+1)(x-3)(x+2)=0 \rightarrow$ **$x = -1$, 3 or -2**.

CHAPTER 7 — Analytical geometry

7.1 Revision: distance

1. $d = \sqrt{25} =$ **5**.
2. $d = \sqrt{100} =$ **10**.
3. $d = \sqrt{25} =$ **5**.
4. $d = \sqrt{25} =$ **5**.
5. $d = \sqrt{25} =$ **5**.
6. $d = \sqrt{25} =$ **5** km.
7. $d = \sqrt{25} =$ **5** km.
8. $d = \sqrt{625} =$ **25** km.
9. $d = \sqrt{25} =$ **5** km.
10. $d = \sqrt{169} =$ **13** km.

7.2 Equation of a circle

1. $x^2 + y^2 = 25$.
2. $x^2 + y^2 = 9$.
3. $r = 6$.
4. $(x - 2)^2 + (y + 3)^2 = 16$.
5. Centre $(1, -2)$, radius 7.
6. $x^2 + y^2 = 64$.
7. $6^2 + 8^2 = 100 = 10^2 \rightarrow$ it is exactly on the boundary, so yes, just in range.
8. $(x - 2)^2 + (y + 3)^2 = 25$.
9. $(x-3)^2+(y+2)^2=25 \rightarrow$ centre $(3, -2)$, radius 5.
10. $r^2 = 5^2+12^2 = 169 \rightarrow x^2 + y^2 = 169$.

7.3 Tangent to a circle

1. Exactly one.
2. 90° (they are perpendicular).
3. Perpendicular → −1/2.
4. m = 4/3.
5. Perpendicular → −3/4.
6. m(tan)=−¾ → y − 4 = −¾(x − 3) → **y = −¾x + 25/4**.
7. m(radius)=−4/3 → m(tan)=¾ → **y − 4 = ¾(x + 3)**.
8. m(radius)=3 → m(tan)=−⅓ → **y − 3 = −⅓(x − 1)**.
9. Distance from O to the line = |5√2|/√2 = 5 = r → **yes, it is a tangent**.
10. m(radius) = (5−1)/(5−2) = 4/3 → m(tan) = **−3/4**.

End of Chapter 7 — Mixed Exercise

1. d = √25 = **5**.
2. $x^2 + y^2 = 9$.
3. Perpendicular → −1/2.
4. d = √25 = **5**.
5. Centre (1, −2), radius 7.
6. m(tan)=−¾ → y − 4 = −¾(x − 3) → **y = −¾x + 25/4**.
7. d = √25 = **5** km.
8. $(x - 2)^2 + (y + 3)^2 = 25$.
9. Distance from O to the line = |5√2|/√2 = 5 = r → **yes, it is a tangent**.
10. d = √169 = **13** km.

CHAPTER 8 — Euclidean geometry

8.1 Revision: cyclic quadrilateral

1. 180° − 80° = **100°**.
2. **85°**.
3. **x = 70°**.
4. **90°** (angle in a semicircle).
5. Half → **70°**.
6. 180° − 88° = **92°**.
7. ∠R = 110°, ∠S = 80°.
8. **90°** (angle in a semicircle).
9. Equal → **38°**.
10. 2x + (x+30) = 180 → 3x = 150 → **x = 50°**.

8.2 Ratio and proportion (intercept theorem)

1. A line parallel to one side of a triangle divides the other two sides proportionally.
2. 4/2 = 6/EC → **EC = 3**.
3. 3/6 = 2/EC → **EC = 4**.
4. DE ∥ BC (the converse).
5. 5/8 = 10/AC → **AC = 16**.
6. 3/6 = 2/x → **x = 4 units**.
7. 2/4 = 6/EC → **EC = 12 m**.
8. **1 : 3**.
9. It is parallel to it and half its length.
10. 2x/(x+2) = 6/4 → 8x = 6x + 12 → **x = 6**.

8.3 Sum of interior angles of polygon

1. 180°.
2. **360°**.
3. **540°**.
4. **720°**.
5. 720/6 = **120°**.
6. Sum = 1080°; each = 135° → **1080°, 135°**.
7. 540/5 = **108°**.
8. 180 − 150 = 30 exterior → 360/30 = **12 sides**.
9. (n−2)·180 = 1440 → **n = 10**.
10. (n−2)180/n = 108 → **n = 5**.

8.4 Triangles: angle sum

1. **50°**.
2. **55°**.
3. **60°**.
4. **45°** each.
5. 6x = 180 → **x = 30°**.
6. (180 − 40)/2 = **70°**.
7. 120 − 50 = **70°**.
8. 5/10 · 180 = **90°**.
9. 4x + 20 = 180 → **x = 40°**.
10. (180 − 80)/2 = **50°**.

8.5 Similarity (ratio of areas)

1. **4 : 1**.
2. **9 : 1**.
3. **4 : 1**.
4. Equal corresponding angles ⇒ the triangles are similar.
5. **25 : 4**.
6. Area ratio = 2^2 → **4 times larger**.
7. 12 × 4 = **48 cm²**.
8. $(6/9)^2 \times 81$ = **36 m²**.
9. $50^2 : 1$ = **2500 : 1**.
10. $(20/30)^2$ = **4 : 9**.

8.6 Pythagorean theorem

1. **5**.
2. **10**.
3. **13**.
4. √(169 − 25) = **12**.
5. $7^2 + 24^2 = 625 = 25^2$ → **yes**.
6. √(100 − 64) = **6 m**.
7. $\sqrt{80^2 + 60^2} = \sqrt{10000}$ = **100 cm**.
8. $\sqrt{1.5^2 + 2^2} = \sqrt{6.25}$ = **2.5 m**.
9. $\sqrt{100^2 + 75^2} = \sqrt{15625}$ = **125 m**.
10. $\sqrt{9^2 + 12^2} = \sqrt{225}$ = **15 km**.

End of Chapter 8 — Mixed Exercise

1. 180° − 80° = **100°**.
2. 4/2 = 6/EC → **EC = 3**.
3. **540°**.
4. **45°** each.
5. **25 : 4**.
6. √(100 − 64) = **6 m**.

7. ∠R = 110°, ∠S = 80°.
8. **1 : 3**.
9. $(n-2)\cdot 180 = 1440 \rightarrow$ **n = 10**.
10. $(180 - 80)/2 =$ **50°**.

CHAPTER 9 — Statistics

9.1 Revision: mean

1. **6**.
2. **8.4**.
3. **6**.
4. **15**.
5. **18**.
6. sum = 360, n = 5 → mean = **72**.
7. sum = 35, n = 7 → mean = **5 mm**.
8. sum = 2250, n = 5 → mean = **R450**.
9. $(50+55+60+x)/4 = 56 \rightarrow 165 + x = 224 \rightarrow$ **x = 59 kg**.
10. $6\cdot 45 - 5\cdot 42 = 270 - 210 =$ **60 runs**.

9.2 Curve fitting (least squares)

1. The sum of the squared vertical distances from the points to the line.
2. $\hat{y} = a + bx$.
3. The gradient (slope).
4. **14**.
5. **7**.
6. $30 + 8\cdot 5 =$ **70%**.
7. $20 + 15\cdot 5 =$ **95 units**.
8. The mean point $(\bar{x}, \bar{y})$.
9. $a = 12 - 2\cdot 4 =$ **4**.
10. It is far outside the data range; the linear trend may not hold (extrapolation).

9.3 Correlation coefficient r

1. $-1 \le r \le 1$.
2. A perfect positive linear correlation.
3. A perfect negative linear correlation.
4. No linear correlation.
5. A strong positive linear correlation.
6. A strong positive linear correlation — more study tends to go with higher marks.
7. A strong negative linear correlation — hotter days, fewer heaters sold.
8. Essentially no linear relationship between them.
9. $r = -0.8$ (closer to ± 1).
10. No — correlation does not imply causation.

9.3 Correlation interpretation

1. Very strong positive linear correlation.
2. Weak negative linear correlation.
3. Essentially no linear correlation.
4. $r \approx 0$.
5. $0.95 > -0.7 > 0.5$.
6. Negative and fairly strong (close to -1).
7. Positive, close to 1.
8. Yes — they may have a non-linear relationship.
9. A lurking variable (hot weather) likely drives both; correlation ≠ causation.
10. Fuel use and engine size ($r = 0.92$).

End of Chapter 9 — Mixed Exercise

1. **6**.
2. $\hat{y} = a + bx$.
3. A perfect negative linear correlation.
4. $r \approx 0$.
5. **18**.
6. 30 + 8·5 = **70%**.
7. A strong negative linear correlation — hotter days, fewer heaters sold.
8. Yes — they may have a non-linear relationship.
9. $(50+55+60+x)/4 = 56 \rightarrow 165 + x = 224 \rightarrow$ **x = 59 kg**.
10. It is far outside the data range; the linear trend may not hold (extrapolation).

CHAPTER 10 — Probability

10.1 Revision: P of single event

1. **1/6**.
2. **1/2**.
3. **1/2**.
4. **1/4**.
5. **0.7**.
6. 5/10 = **1/2**.
7. 12/30 = **2/5**.
8. 3/8.
9. 8/12 = **2/3**.
10. **1/4**.

10.2 Probability identities

1. $P(A \cap B)$.
2. **0.7**.
3. 0.
4. $P(A)\cdot P(B)$.
5. **0.3**.
6. 0.5 + 0.4 − 0.2 = **0.7**.
7. 40 + 30 − 18 = **52%**.
8. **0.75**.
9. 0.7 · 0.6 = **0.42**.
10. $0.28 = 0.7\cdot P(B) \rightarrow$ **0.4**.

10.3 Tools: tree diagram

1. 2^3 = **8**.
2. HH, HT, TH, TT.
3. 6^2 = **36**.
4. 2·6 = **12**.
5. 3/8.
6. 3^2 = **9**.
7. (2/3)(1/2) = **1/3**.
8. (2/3)(2/3) = **4/9**.
9. $(1/2)^4$ = **1/16**.
10. 0.4 · 0.4 = **0.16**.

10.4 Fundamental counting principle

1. **15**.
2. **24**.
3. 10^4 = **10 000**.
4. 26^3 = **17 576**.
5. 5! = **120**.
6. $26^3 \cdot 10^3$ = **17 576 000**.
7. 3·4·2 = **24**.
8. 26 · 10 · 9 = **2 340**.
9. 9·10·5 = **450**.
10. Treat the pair as one block: 4!·2! = **48**.

10.5 Factorial notation

1. **6**.
2. **24**.
3. **120**.
4. **1**.
5. **30**.
6. 6! = **720**.
7. 7·6 = **42**.
8. **n + 1**.
9. **28**.
10. n(n−1) = 20 → **n = 5**.

10.6 Counting: permutations nPr

1. **20**.
2. **120**.
3. **42**.
4. **24**.
5. **72**.
6. ${}^{8}P_{3}$ = **336**.
7. ${}^{10}P_{4}$ = **5040**.
8. ${}^{7}P_{3}$ = **210**.
9. ${}^{5}P_{3}$ = **60**.
10. n(n−1) = 30 → **n = 6**.

10.6 Counting: combinations nCr

1. **10**.
2. **20**.
3. **21**.
4. **1**.
5. **84**.
6. ${}^{10}C_{4}$ = **210**.
7. ${}^{6}C_{2}$ = **15**.
8. ${}^{5}C_{2} \cdot {}^{4}C_{2}$ = 10·6 = **60**.
9. ${}^{8}C_{3}$ = **56**.
10. n(n−1)/2 = 15 → **n = 6**.

10.7 Probability with counting

1. **1/100**.
2. **1/10 000**.
3. 4/52 = **1/13**.
4. $1/{}^{5}C_{2}$ = **1/10**.
5. ${}^{3}C_{2}/{}^{5}C_{2}$ = **3/10**.
6. ${}^{4}C_{3}/{}^{6}C_{3}$ = 4/20 = **1/5**.

7. $4!/5! =$ **$1/5$**.
8. $(4!\cdot 2!)/5! =$ **$2/5$**.
9. $({}^3C_2\cdot {}^3C_1)/{}^6C_3 = (3\cdot 3)/20 =$ **$9/20$**.
10. 1/□${}^6P_5 =$ **$1/7893600$**.

End of Chapter 10 — Mixed Exercise

1. **$1/6$**.
2. **0.7**.
3. $6^2 =$ **36**.
4. $26^3 =$ **17 576**.
5. **30**.
6. ${}^8P_3 =$ **336**.
7. ${}^6C_2 =$ **15**.
8. $(4!\cdot 2!)/5! =$ **$2/5$**.
9. $8/12 =$ **$2/3$**.
10. $0.28 = 0.7\cdot P(B) \rightarrow$ **0.4**.

CHAPTER 7 — Exponents

12.E1 Exponents revision (advanced laws)

1. **a^8**.
2. **x^5**.
3. **a^{12}**.
4. **2**.
5. **$8x^6y^3$**.
6. $2^6\cdot 2^6 =$ **2^{12} (= 4096)**.
7. **a^3b**.
8. **$2000\cdot 2^x$**.
9. $3^4\cdot 3^3 =$ **3^7**.
10. $3\cdot 2^{(x-(x-1))} =$ **6**.

12.E2 Solving exponential equations

1. **$x = 3$**.
2. **$x = 4$**.
3. **$x = 5$**.
4. **$x = 0$**.
5. $2^{(2x)} = 2^3 \rightarrow$ **$x = 3/2$**.
6. **$x = 5$ hours**.
7. $x = \log 10 / \log 3 \approx$ **2.1 years**.
8. $x + 1 = 5 \rightarrow$ **$x = 4$**.
9. Let $y = 2^x$: $y^2 - 5y + 4 = 0 \rightarrow y = 1$ or $4 \rightarrow$ **$x = 0$ or 2**.
10. $y = 3^x$: $(y-1)(y-3) = 0 \rightarrow$ **$x = 0$ or 1**.

12.E3 Surds and rationalising

1. **$5\sqrt{2}$**.
2. **$4\sqrt{3}$**.
3. $2\sqrt{2} + 3\sqrt{2} =$ **$5\sqrt{2}$**.
4. $\sqrt{16} =$ **4**.
5. $2\sqrt{3} - 3\sqrt{3} =$ **$-\sqrt{3}$**.
6. $\sqrt{50} =$ **$5\sqrt{2}$ m**.
7. **$\sqrt{2}/2$**.
8. **$\sqrt{3}$**.
9. $\times(2 - \sqrt{3})$: $(2 - \sqrt{3})/(4 - 3) =$ **$2 - \sqrt{3}$**.
10. $7 - 2 =$ **5**.

End of Chapter 7 — Mixed Exercise

1. $\mathbf{a^8}$.
2. $\mathbf{x = 4}$.
3. $2\sqrt{2} + 3\sqrt{2} = \mathbf{5\sqrt{2}}$.
4. **2**.
5. $2^{(2x)} = 2^3 \rightarrow \mathbf{x = 3/2}$.
6. $\sqrt{50} = \mathbf{5\sqrt{2}}$ **m**.
7. $\mathbf{a^3b}$.
8. $x + 1 = 5 \rightarrow \mathbf{x = 4}$.
9. $\times(2 - \sqrt{3})$: $(2 - \sqrt{3})/(4 - 3) = \mathbf{2 - \sqrt{3}}$.
10. $3 \cdot 2^{(x-(x-1))} = \mathbf{6}$.

CHAPTER 8 — Number Patterns

12.P1 Arithmetic sequences

1. **19**.
2. $\mathbf{d = 4}$.
3. **29**.
4. **59**.
5. $\mathbf{T_n = 5n - 4}$.
6. $a = 20$, $d = 4 \rightarrow T_{10} =$ **56 seats**.
7. $a = 3$, $d = 4 \rightarrow T_{12} =$ **47 logs**.
8. $a = 8000$, $d = 500 \rightarrow T_6 =$ R**10500**.
9. $4 + (n-1)3 = 100 \rightarrow \mathbf{n = 33}$.
10. $d = (27-11)/4 = 4$, $a = 3 \rightarrow \mathbf{a = 3, d = 4}$.

12.P2 Quadratic sequences

1. First diffs 3, 5, 7 → second diff **2**.
2. $2a = 2 \rightarrow \mathbf{a = 1}$.
3. $2a = 2 \rightarrow \mathbf{a = 1}$.
4. Diffs 5, 7, 9 → next diff 11 → **35**.
5. $\mathbf{T_n = n^2}$.
6. $\mathbf{T_{10} = 100}$ **dots**.
7. $\mathbf{T_n = n^2 + n}$.
8. 2, 6, 13, 23 → $\mathbf{T_4 = 23}$.
9. Diffs 5, 9, 13 → next 17 → $\mathbf{T_5 = 47}$.
10. $(n-2)(n-3) = 0 \rightarrow$ **n = 2 or 3**.

12.P3 Geometric sequences

1. **54**.
2. $\mathbf{r = 2}$.
3. **625**.
4. **192**.
5. $\mathbf{T_n = 2^{(n+1)}}$.
6. $a = 16$, $r = ½ \rightarrow T_6 =$ **0.5 m**.
7. $3 \cdot 2^{(n-1)} = 96 \rightarrow 2^{(n-1)} = 32 \rightarrow \mathbf{n = 6}$.
8. $a = 200000$, $r = 0.8 \rightarrow T_4 =$ R**102400**.
9. $r^2 = 54/6 = 9 \rightarrow \mathbf{r = 3}$.
10. $(x+3)^2 = x(x+9) \rightarrow 9 = 3x \rightarrow \mathbf{x = 3}$.

12.P4 Finite arithmetic series

1. **55**.
2. **110**.
3. $a = 2$, $d = 3$, $n = 7 \rightarrow$ **77**.
4. **820**.
5. **1275**.
6. $a = 20$, $d = 2$, $n = 15 \rightarrow S =$ **510 seats**.
7. $a = 100$, $d = 20$, $n = 12 \rightarrow S =$ R**2520**.
8. $n(3 + 3n)/2 = 165 \rightarrow n^2 + n - 110 = 0 \rightarrow$ $\mathbf{n = 10}$.
9. **390**.
10. $T_5 = S_5 - S_4 = 65 - 44 =$ **21**.

12.P5 Finite geometric series

1. **63**.
2. **45**.
3. **200**.
4. **364**.
5. **124**.
6. $a = 2$, $r = 2$, $n = 6 \rightarrow S =$ **126 messages**.
7. $1 + 2 + \ldots$ (10 terms) = **1023 grains**.
8. **1.88 m** (= 15/8).
9. $2(3^n - 1)/2 = 728 \rightarrow 3^n = 729 \rightarrow$ $\mathbf{n = 6}$.
10. $3(1 + r + r^2) = 21 \rightarrow r^2 + r - 6 = 0 \rightarrow$ $\mathbf{r = 2}$.

12.P6 Infinite geometric series

1. $|r| < 1$.
2. $4/(1 - \frac{1}{2}) =$ **8**.
3. $9/(2/3) =$ **13.5**.
4. **12**.
5. $r = 2$, $|r| > 1 \rightarrow$ **no, it diverges**.
6. $2/(1 - \frac{3}{4}) =$ **8 m**.
7. $40/(1 - 0.8) =$ **200 cm**.
8. $0.7/(1 - 0.1) =$ **7/9**.
9. $8/(1 - r) = 16 \rightarrow$ $\mathbf{r = \frac{1}{2}}$.
10. Converges for $|x| < 1$ to $\mathbf{1/(1 - x)}$.

Past Paper — Number Patterns (quadratic sequence)

1. 3, 8, 15.
2. **4**.
3. $2a = 4 \rightarrow$ $\mathbf{a = 2}$.
4. $\mathbf{T_n = 2n^2 - n + 2}$.
5. **192**.
6. $2n^2 - n + 2 = 71 \rightarrow 2n^2 - n - 69 = 0 \rightarrow$ $\mathbf{n = 6}$.
7. $\mathbf{T_n = n^2 + n}$.
8. $T_n - T_{n-1} =$ $\mathbf{4n - 3}$.
9. $4, 7, 14, 25 \rightarrow$ $\mathbf{T_4 = 25}$.
10. $4n - 3 > 40 \rightarrow n \geq 11 \rightarrow T_{11} =$ **233**.

End of Chapter 8 — Mixed Exercise

1. **19**.
2. $2a = 2 \rightarrow$ $\mathbf{a = 1}$.
3. **625**.
4. **820**.
5. **124**.
6. $2/(1 - \frac{3}{4}) =$ **8 m**.

7. **$T_n = n^2 + n$**.
8. a = 8000, d = 500 → T_6 = R**10500**.
9. Diffs 5, 9, 13 → next 17 → **$T_5 = 47$**.
10. $(x+3)^2 = x(x+9)$ → 9 = 3x → **x = 3**.

CHAPTER 9 — Functions

12.F1 Straight Line function

1. **m = 2**.
2. **(0, −9)**.
3. **(3, 0)**.
4. x = 2y − 4 → **y = (x + 4)/2**.
5. **y = (6 − x)/2**.
6. x = (y − 350)/250 → **hours = (cost − 350)/250**.
7. x = 18d → **rand = 18 × dollars**.
8. y = x.
9. On y = x: 2x − 4 = x → **(4, 4)**.
10. x = (y − 12)/8 → for y = 60: x = 48/8 = **6 km**.

12.F2 Quadratic function

1. Otherwise it is not one-to-one (fails the horizontal line test).
2. **$y = \sqrt{x}$**.
3. **$y = -\sqrt{x}$**.
4. y ≥ 0.
5. **(9, 3)**.
6. $t = \sqrt{(h/5)}$ → **$t = \sqrt{(h/5)}$ s**.
7. $r = \sqrt{(A/\pi)}$ → **$r = \sqrt{(A/\pi)}$**.
8. $x = 2y^2$ → **$y = \sqrt{(x/2)}$**.
9. No — $y = \pm\sqrt{x}$ is one-to-many.
10. $x = (y - 1)^2$ → **$y = \sqrt{x} + 1$**.

12.F3 Hyperbolic function

1. x = 0 and y = 0.
2. **y = 4/x** (self-inverse).
3. Swapping x and y returns the same equation.
4. x = 3 and y = 0.
5. x = 0 and y = 2.
6. A hyperbola; asymptotes x = 0 and y = 0 (more workers → less time).
7. 240/8 = 30 h; and 240/x = 10 → **x = 24 workers**.
8. **y = 2/x**.
9. $x^2 = 4$ → **(2, 2) and (−2, −2)**.
10. x = 3/y + 1 → **y = 3/(x − 1)**.

12.F4 Exponential function

1. **3**.
2. **3**.
3. **$y = \log_2 x$**.
4. **x = 3**.
5. y = 0 (the x-axis).
6. $t = \log_2 N$ → **$t = \log_2 N$ hours**.
7. t = log 1000 / log 2 ≈ **9.97 hours**.
8. x = log 20 / log 5 ≈ **1.86 years**.
9. $x = 3 \cdot 2^y$ → **$y = \log_2(x/3)$**.
10. (0, 1); inverse cuts at (1, 0).

End of Chapter 9 — Mixed Exercise

1. **$m = 2$**.
2. **$y = \sqrt{x}$**.
3. Swapping x and y returns the same equation.
4. **$x = 3$**.
5. **$y = (6 - x)/2$**.
6. $t = \sqrt{(h/5)} \rightarrow$ **$t = \sqrt{(h/5)}$ s**.
7. $240/8 = 30$ h; and $240/x = 10 \rightarrow$ **$x = 24$ workers**.
8. $x = \log 20 / \log 5 \approx$ **1.86 years**.
9. On $y = x$: $2x - 4 = x \rightarrow$ **(4, 4)**.
10. $x = (y - 1)^2 \rightarrow$ **$y = \sqrt{x} + 1$**.

CHAPTER 12 — Trigonometry

12.1 Compound angle identities

1. $\sin A \cos B + \cos A \sin B$.
2. $\cos A \cos B - \sin A \sin B$.
3. $\cos A \cos B + \sin A \sin B$.
4. $\sin(90°) =$ **1**.
5. $\cos(60°) =$ **1/2**.
6. **$(\sqrt{6} + \sqrt{2})/4$**.
7. **$(\sqrt{6} + \sqrt{2})/4$**.
8. $\cos 75° = \cos(45° + 30°) =$ **$(\sqrt{6} - \sqrt{2})/4$**.
9. **$\sin\theta$**.
10. Expanding, the $\pm\sin A \sin B$ terms cancel → **$2\cos A \cos B$**.

12.2 Double angle identities

1. $2 \sin\theta \cos\theta$.
2. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
3. $2\cdot(3/5)(4/5) =$ **24/25**.
4. $1 - 2(3/5)^2 =$ **7/25**.
5. $\sin 30° =$ **1/2**.
6. $\cos\theta = 15/17 \rightarrow \sin 2\theta = 2\cdot(8/17)(15/17) =$ **240/289**.
7. $1 - 2(8/17)^2 =$ **161/289**.
8. $\cos 70°$.
9. $2\sin\theta\cos\theta/(2\cos^2\theta) =$ **$\tan\theta$**.
10. $2(12/13)^2 - 1 =$ **119/169**.

12.3 Solving equations with identities

1. **$\theta = 30°$ or $150°$**.
2. **$\theta = 90°$ or $270°$**.
3. **$\theta = 45°$ or $225°$**.
4. $\cos\theta = \sqrt{3}/2 \rightarrow$ **$\theta = 30°$ or $330°$**.
5. $2\theta = 60° \rightarrow$ **$\theta = 30°$**.
6. **$\theta = 30°$ or $150°$**.
7. $\sin\theta(2\cos\theta - 1) = 0 \rightarrow$ **$\theta = 0°, 180°, 360°, 60°, 300°$**.
8. $(2\cos\theta + 1)(\cos\theta - 1) = 0 \rightarrow$ **$\theta = 0°, 360°, 120°, 240°$**.
9. $\sin\theta(2\sin\theta - 1) = 0 \rightarrow$ **$\theta = 0°, 180°, 360°, 30°, 150°$**.
10. $(2\cos\theta - 1)(\cos\theta + 2) = 0 \rightarrow \cos\theta = ½ \rightarrow$ **$\theta = 60°$ or $300°$**.

12.4 Applications of trigonometric functions

1. $\sin 60° = 3/L \rightarrow L = 3/\sin 60° \approx$ **3.46 m**.
2. $50 \tan 30° \approx$ **28.87 m**.
3. $10/\tan 45° =$ **10 m**.
4. $b = 8 \sin 40°/\sin 60° \approx$ **5.94**.
5. $½·6·8·\sin 30° =$ **12 sq units**.
6. $40/\tan 25° \approx$ **85.78 m**.
7. $a^2 = 49 + 81 - 2·7·9·\cos 50° \rightarrow a \approx$ **6.93**.
8. $60 \tan 35° \approx$ **42.01 m**.
9. $\cos A = (49 + 64 - 25)/(2·7·8) \rightarrow A \approx$ **38.2°**.
10. Cosine rule, included angle 120°: $\sqrt{(200^2 + 150^2 - 2·200·150·\cos 120°)} \approx$ **304.1 km**.

End of Chapter 12 — Mixed Exercise

1. $\sin A \cos B + \cos A \sin B$.
2. $\cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$.
3. **θ = 45° or 225°**.
4. $b = 8 \sin 40°/\sin 60° \approx$ **5.94**.
5. $\cos(60°) =$ **1/2**.
6. $\cos\theta = 15/17 \rightarrow \sin 2\theta = 2·(8/17)(15/17) =$ **240/289**.
7. $\sin\theta(2\cos\theta - 1) = 0 \rightarrow$ **θ = 0°, 180°, 360°, 60°, 300°**.
8. $60 \tan 35° \approx$ **42.01 m**.
9. **$\sin\theta$**.
10. $2(12/13)^2 - 1 =$ **119/169**.

CHAPTER 11 — Calculus

12.C1 Differentiation from first principles

1. $f'(x) = \lim(h\rightarrow 0)\ [f(x+h) - f(x)]/h$.
2. **0**.
3. **3**.
4. $[(x+h)^2 - x^2]/h = 2x + h \rightarrow$ **2x**.
5. **2x**.
6. **4x**.
7. **2x − 3**.
8. $[(x+h)^3 - x^3]/h = 3x^2 + 3xh + h^2 \rightarrow$ **$3x^2$**.
9. **−2x**.
10. **$-1/x^2$**.

12.C2 Power rule (differentiating x^n)

1. **$4x^3$**.
2. **$7x^6$**.
3. **10x**.
4. **$15x^4$**.
5. **0**.
6. $s'(t) =$ **8t m/s**.
7. **−2x^(−3)**.
8. **$1/(2\sqrt{x})$**.
9. $V'(t) =$ **$-8/t^3$**.
10. **$3\sqrt{x}$**.

12.C3 Differentiating polynomials

1. **$6x + 2$**.
2. **$3x^2 - 8x + 7$**.
3. **$10x - 2$**.
4. **$4x^3 - 2x$**.
5. $x^2 - x - 6 \rightarrow$ **$2x - 1$**.
6. $h'(t) =$ **$20 - 10t$ m/s**.
7. $h'(1) = 20 - 10 =$ **10 m/s**.
8. $P'(x) =$ **$3x^2 - 3$**.
9. $4x^2 - 4x + 1 \rightarrow$ **$8x - 4$**.
10. $f'(x) = 3x^2 - 3 \rightarrow f'(2) =$ **9**.

12.C4 Tangents and gradients

1. $f'(x) = 2x \rightarrow$ **6**.
2. $f'(x) = 3x^2 \rightarrow$ **12**.
3. $2x - 6 = 0 \rightarrow$ **$x = 3$**.
4. **$y = 2x - 1$**.
5. **$y = 6x - 9$**.
6. $f'(2) =$ **4**.
7. $f(2) = 2, f'(2) = 9 \rightarrow$ **$y = 9x - 16$**.
8. $2x = 8 \rightarrow$ **(4, 16)**.
9. tangent $m = 2 \rightarrow$ normal $m = -½ \rightarrow$ **$y = -½x + 3/2$**.
10. $3x^2 - 12 = 0 \rightarrow x = \pm 2 \rightarrow$ **(2, −16) and (−2, 16)**.

12.C5 Second derivative and concavity

1. **$6x$**.
2. **$12x^2$**.
3. **$6x - 12$**.
4. **$12x$**.
5. **$x = 2$**.
6. $s''(t) =$ **$6t - 12$**.
7. $6t - 12 = 0 \rightarrow$ **$t = 2$ s**.
8. $f''(x) = 6x - 6 > 0 \rightarrow$ **$x > 1$**.
9. $f''(x) = 2 > 0 \rightarrow$ **minimum**.
10. $f''(x) = -2 < 0 \rightarrow$ **maximum**.

12.C6 Cubic polynomials

1. $f(1) = 1 - 6 + 11 - 6 = 0 \rightarrow$ **yes**.
2. **$(x - 1)(x - 2)(x - 3)$**.
3. **$x = 0, 2, -2$**.
4. **(0, −5)**.
5. $3x^2 - 3 = 0 \rightarrow$ **$x = \pm 1$**.
6. $f(2) = 0 \rightarrow (x - 2)(x^2 - x - 2) =$ **$(x - 2)^2(x + 1)$**.
7. $f'(x) = 3x^2 - 12x + 9 = 0 \rightarrow x = 1, 3 \rightarrow$ **(1, 4) and (3, 0)**.
8. $(x - 1)^2(x + 2) \rightarrow$ **2 distinct real roots** ($x = 1$ repeated, $x = -2$).
9. **$(x + 1)(x - 2)(x - 3) = x^3 - 4x^2 + x + 6$**.
10. $8 - 2k + 2 = 0 \rightarrow$ **$k = 5$**.

12.C7 Sketching cubics

1. **(0, 2)**.
2. $x = \pm 1 \rightarrow$ **(−1, 2) and (1, −2)**.
3. $f''(x) = 6x$: $x = -1$ is a max, $x = 1$ is a min.
4. $6x = 0 \rightarrow$ **(0, 0)**.
5. $x(x - 3)^2 = 0 \rightarrow$ **$x = 0$ and $x = 3$ (double)**.
6. $f'(x) = 3x^2 \geq 0 \rightarrow$ **increasing** (with a stationary point at $x = 0$).

7. $x = \pm 1$; $x = 1$ is a max (value 2), $x = -1$ is a min (value -2).
8. **(1, 4) and (3, 0)**.
9. $\mathbf{f(x) = x^3 - 3x}$.
10. $3x^2 - 3 > 0 \rightarrow \mathbf{x < -1 \text{ or } x > 1}$.

12.C8 Optimisation

1. $A'(x) = 6 - 2x = 0 \rightarrow \mathbf{x = 3}$.
2. $A(3) = \mathbf{9}$.
3. $f'(x) = 2x - 6 = 0 \rightarrow x = 3 \rightarrow \mathbf{1}$.
4. $P = x(20 - x)$, max at $x = 10 \rightarrow$ **product 100**.
5. $x + y = 10$, $A = x(10 - x)$, max at $x = 5 \rightarrow$ **5 m × 5 m**.
6. **25 m²**.
7. $A = x(100 - 2x)$; $A' = 100 - 4x = 0 \rightarrow x = 25 \rightarrow$ **25 m × 50 m**.
8. $25 \times 50 =$ **1 250 m²**.
9. $V'(x) = 0 \rightarrow 3x^2 - 20x + 25 = 0 \rightarrow \mathbf{x = 5/3}$.
10. Write surface area in one variable using the volume constraint, set $dA/dr = 0$, and solve for r.

12.C9 Cubic Quest (interactive game)

1. $\mathbf{3x^2}$.
2. $\mathbf{x = \pm 1}$.
3. $\mathbf{x = -1}$ (value 2).
4. $\mathbf{x(x - 1)(x + 1)}$.
5. $\mathbf{x = 0, 2, -2}$.
6. $\mathbf{6x - 12}$.
7. $\mathbf{(2, -11)}$.
8. $\mathbf{x < -1 \text{ or } x > 1}$.
9. $f(1) = 1$, $f'(1) = 3 \rightarrow \mathbf{y = 3x - 2}$.
10. **Two**.

End of Chapter 11 — Mixed Exercise

1. $f'(x) = \lim(h\rightarrow 0)\ [f(x+h) - f(x)]/h$.
2. $\mathbf{7x^6}$.
3. $\mathbf{10x - 2}$.
4. $\mathbf{y = 2x - 1}$.
5. $\mathbf{x = 2}$.
6. $f(2) = 0 \rightarrow (x - 2)(x^2 - x - 2) = \mathbf{(x - 2)^2(x + 1)}$.
7. $x = \pm 1$; $x = 1$ is a max (value 2), $x = -1$ is a min (value -2).
8. $25 \times 50 =$ **1 250 m²**.
9. $f(1) = 1$, $f'(1) = 3 \rightarrow \mathbf{y = 3x - 2}$.
10. $\mathbf{-1/x^2}$.

CHAPTER 12 — Past Papers — Algebra

Past Papers — Algebra: Equations

1. $(x - 2)(x - 3) = 0 \rightarrow \mathbf{x = 2 \text{ or } 3}$.
2. $\mathbf{x = \pm 3}$.
3. $\mathbf{x = 0 \text{ or } x = -4/3}$.
4. $(x + 4)(x - 2) = 0 \rightarrow \mathbf{x = -4 \text{ or } 2}$.
5. $x = (4 \pm \sqrt{8})/4 \rightarrow \mathbf{x \approx 1.71 \text{ or } 0.29}$.
6. $5t(4 - t) = 0 \rightarrow$ **t = 0 s or t = 4 s**.
7. $w(w + 3) = 40 \rightarrow w^2 + 3w - 40 = 0 \rightarrow (w + 8)(w - 5) = 0 \rightarrow$ **w = 5 m**.
8. Squaring: $x^2 - 9x + 18 = 0 \rightarrow x = 3$ or 6; $x = 6$ makes $4 - x$ negative, so $\mathbf{x = 3}$.
9. $y = 2 - 3x \rightarrow 7x^2 - 12x + 5 = 0 \rightarrow$ **(1, −1) and (5/7, −1/7)**.
10. Let $y = x^2$: $(y - 1)(y - 4) = 0 \rightarrow \mathbf{x = \pm 1, \pm 2}$.

Past Papers — Algebra: Inequalities

1. $\mathbf{x > 3}$.
2. $\mathbf{x \leq 2}$.
3. $\mathbf{x < 1}$ **or** $\mathbf{x > 3}$.
4. $\mathbf{-2 < x < 4}$.
5. $\mathbf{-3 \leq x \leq 3}$.
6. $\mathbf{x < 2}$ **or** $\mathbf{x > 6}$.
7. $x(4 - x) > 0 \rightarrow \mathbf{0 < x < 4}$.
8. $\mathbf{x \leq 1}$ **or** $\mathbf{x \geq 3}$.
9. $x(x - 4) \geq 0 \rightarrow \mathbf{x \leq 0}$ **or** $\mathbf{x \geq 4}$.
10. $\mathbf{x \leq -1}$ **or** $\mathbf{x > 2}$.

Past Papers — Algebra: Proofs & Identities

1. $\mathbf{a^2 - b^2}$.
2. $\mathbf{a^2 + 2ab + b^2}$.
3. $\mathbf{a^2 - 2ab + b^2}$.
4. Add: $2r = a + b \rightarrow \mathbf{r = (a + b)/2}$.
5. Subtract: $4s = a - b \rightarrow \mathbf{s = (a - b)/4}$.
6. $n + (n + 1) = 2n + 1$, which is odd.
7. $(x + 2)(x - 2) = x^2 - 4 \rightarrow$ it is exactly 4 m^2 less.
8. Expanding: $(a^2+2ab+b^2) - (a^2-2ab+b^2) = \mathbf{4ab}$.
9. $(x - 3)^2 + 1 \geq 1 > 0$ for all x.
10. $(2n)(2n + 2) = 4n(n + 1)$; $n(n + 1)$ is even $\rightarrow$ divisible by 8.

End of Chapter 12 — Mixed Exercise

1. $(x - 2)(x - 3) = 0 \rightarrow \mathbf{x = 2}$ **or 3**.
2. $\mathbf{x \leq 2}$.
3. $\mathbf{a^2 - 2ab + b^2}$.
4. $(x + 4)(x - 2) = 0 \rightarrow \mathbf{x = -4}$ **or 2**.
5. $\mathbf{-3 \leq x \leq 3}$.
6. $n + (n + 1) = 2n + 1$, which is odd.
7. $w(w + 3) = 40 \rightarrow w^2 + 3w - 40 = 0 \rightarrow (w + 8)(w - 5) = 0 \rightarrow \mathbf{w = 5}$ **m**.
8. $\mathbf{x \leq 1}$ **or** $\mathbf{x \geq 3}$.
9. $(x - 3)^2 + 1 \geq 1 > 0$ for all x.
10. Let $y = x^2$: $(y - 1)(y - 4) = 0 \rightarrow \mathbf{x = \pm 1, \pm 2}$.

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